

# Diffusive KPP Equations with Free Boundaries in Time Almost Periodic Environments: II. Spreading Speeds and Semi-Wave Solutions

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**Abstract.** In this series of papers, we investigate the spreading and vanishing dynamics of time almost periodic diffusive KPP equations with free boundaries. Such equations are used to characterize the spreading of a new species in time almost periodic environments with free boundaries representing the spreading fronts. In the first part of the series, we showed that a spreading-vanishing dichotomy occurs for such free boundary problems (see [16]). In this second part of the series, we investigate the spreading speeds of such free boundary problems in the case that the spreading occurs. We first prove the existence of a unique time almost periodic semi-wave solution associated to such a free boundary problem. Using the semi-wave solution, we then prove that the free boundary problem has a unique spreading speed.

**Keywords.** Diffusive KPP equation, free boundary, time almost periodic environment, spreading-vanishing dichotomy, spreading speed, time almost periodic semi-wave solution, principal Lyapunov exponent.

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# 1 Introduction

This is the second part of a series of papers on the spreading and vanishing dynamics of diffusive equations with free boundaries of the form,

$$\begin{cases} u_t = u_{xx} + uf(t, x, u), & t > 0, 0 < x < h(t) \\ h'(t) = -\mu u_x(t, h(t)), & t > 0 \\ u_x(t, 0) = u(t, h(t)) = 0, & t > 0 \\ h(0) = h_0, u(0, x) = u_0(x), & 0 \leq x \leq h_0, \end{cases} \quad (1.1)$$

where  $\mu > 0$ . We assume that  $f(t, x, u)$  is a  $C^1$  function in  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}$ , and  $u \in \mathbb{R}$ ;  $f(t, x, u) < 0$  for  $u \gg 1$ ;  $f_u(t, x, u) < 0$  for  $u \geq 0$ , and  $f(t, x, u)$  is almost periodic in  $t$  uniformly with respect to  $x \in \mathbb{R}$  and  $u$  in bounded sets of  $\mathbb{R}$  (see (H1), (H2) in section 2 for detail). Here is a typical example of such functions,  $f(t, x, u) = a(t, x) - b(t, x)u$ , where  $a(t, x)$  and  $b(t, x)$  are almost periodic in  $t$  and periodic in  $x \in \mathbb{R}$ , and  $\inf_{t \in \mathbb{R}, x \in \mathbb{R}} b(t, x) > 0$ .

Observe that for given  $h_0 > 0$  and  $u_0$  satisfying

$$u_0 \in C^2([0, h_0]), \quad u_0'(0) = u_0(h_0) = 0, \quad \text{and } u_0 > 0 \text{ in } [0, h_0), \quad (1.2)$$

(1.1) has a (local) solution  $(u(t, \cdot; u_0, h_0), h(t; u_0, h_0))$  with  $u(0, \cdot; u_0, h_0) = u_0(\cdot)$  and  $h(0; u_0, h_0) = h_0$  (see [7]). Moreover, by comparison principle for parabolic equations,  $(u(t, \cdot; u_0, h_0), h(t; u_0, h_0))$  exists for all  $t > 0$  and  $u_x(t, h(t; u_0, h_0); u_0, h_0) < 0$ . Hence  $h(t; u_0, h_0)$  increases as  $t$  increases.

Equation (1.1) with  $f(t, x, u) = u(a - bu)$  and  $a$  and  $b$  being two positive constants was introduced by Du and Lin in [9] to understand the spreading of species. A great deal of previous mathematical investigation on the spreading of species (in one space dimension case) has been based on diffusive equations of the form

$$u_t = u_{xx} + uf(t, x, u), \quad x \in \mathbb{R}, \quad (1.3)$$

where  $f(t, x, u) < 0$  for  $u \gg 1$  and  $f_u(t, x, u) < 0$  for  $u \geq 0$ . Thanks to the pioneering works of Fisher ([12]) and Kolmogorov, Petrowsky, Piscunov ([14]) on the following special case of (1.3)

$$u_t = u_{xx} + u(1 - u), \quad x \in \mathbb{R}, \quad (1.4)$$

(1.1), resp. (1.3), is referred to as diffusive Fisher or KPP equation.

One of the central problems for both (1.1) and (1.3) is to understand their spreading dynamics. For (1.3), this is closely related to spreading speeds and transition fronts of (1.3) and has been widely studied (see [4, 17, 21, 24, 34], etc. for the study in the case that  $f(t, x, u)$  is periodic in  $t$  and/or  $x$ , and see [2, 3, 5, 13, 15, 22, 25, 27, 28, 29, 33, 36], etc. for the study in the case that the dependence of  $f(t, x, u)$  on  $t$  or  $x$  is non-periodic). The spreading dynamics for (1.3) in many cases, including the cases that  $f$  is periodic in  $t$  and  $x$ , is quite well understood. For example, when  $f(t, x, u)$  is periodic in  $t$  and independent of  $x$ , or is independent of  $t$  and periodic in  $x$ , it has been proved that (1.3) has a unique positive periodic solution  $u^*(t, x)$  which

is asymptotically stable with respect to periodic perturbations and has a spreading speed  $c^* \in \mathbb{R}$  in the sense that for any given  $u_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R}^+)$  with non-empty compact support,

$$\begin{cases} \lim_{|x| \leq c' t, t \rightarrow \infty} [u(t, x; u_0) - u^*(t, x)] = 0 & \forall c' < c^* \\ \lim_{|x| \geq c'' t, t \rightarrow \infty} u(t, x; u_0) = 0 & \forall c'' > c^*, \end{cases} \quad (1.5)$$

where  $u(t, x; u_0)$  is the solution of (1.3) with  $u(0, x; u_0) = u_0(x)$  (see [17, 34]).

The spreading property (1.5) for (1.3) in the case that  $f(t, x, u)$  is periodic in  $t$  and independent of  $x$  or independent of  $t$  and periodic in  $x$  implies that spreading always happens for a solution of (1.3) with a positive initial data, no matter how small the positive initial data is. The following strikingly different spreading scenario has been proved for (1.1) in the case that  $f(t, x, u) \equiv f(u)$  (see [6, 9]): it exhibits a spreading-vanishing dichotomy in the sense that for any given positive constant  $h_0$  and initial data  $u_0(\cdot)$  satisfying (1.2), either vanishing occurs (i.e.  $\lim_{t \rightarrow \infty} h(t; u_0, h_0) < \infty$  and  $\lim_{t \rightarrow \infty} u(t, x; u_0, h_0) = 0$ ) or spreading occurs (i.e.  $\lim_{t \rightarrow \infty} h(t; u_0, h_0) = \infty$  and  $\lim_{t \rightarrow \infty} u(t, x; u_0, h_0) = u^*$  locally uniformly in  $x \in \mathbb{R}^+$ , where  $u^*$  is the unique positive solution of  $f(u) = 0$ ). The above spreading-vanishing dichotomy for (1.1) with  $f(t, x, u) \equiv f(u)$  has also been extended to the cases that  $f(t, x, u)$  is periodic in  $t$  or that  $f(t, x, u)$  is independent of  $t$  and periodic in  $x$  (see [7, 8]). The spreading-vanishing dichotomy proved for (1.1) in [6, 7, 8, 9] is well supported by some empirical evidences, for example, the introduction of several bird species from Europe to North America in the 1900s was successful only after many initial attempts (see [18, 32]).

While the spreading dynamics for (1.3) with non-periodic time and/or space dependence has been studied by many people recently (see [2, 3, 5, 13, 15, 22, 25, 27, 28, 29, 33, 36], etc.), there is little study on the spreading dynamics for (1.1) with non-periodic time and space dependence.

The objective of the current series of papers is to investigate the spreading-vanishing dynamics of (1.1) in the case that  $f(t, x, u)$  is almost periodic in  $t$ , that is, to investigate whether the population will successfully establishes itself in the entire space (i.e. spreading occurs), or it fails to establish and vanishes eventually (i.e. vanishing occurs). Roughly speaking, for given  $h_0 > 0$  and  $u_0$  satisfying (1.2), if  $h_\infty = \lim_{t \rightarrow \infty} h(t; u_0, h_0) = \infty$  and for any  $M > 0$ ,  $\liminf_{t \rightarrow \infty} \inf_{0 \leq x \leq M} u(t, x; u_0, h_0) > 0$ , we say *spreading* occurs. If  $h_\infty < \infty$  and  $\lim_{t \rightarrow \infty} u(t, x; u_0, h_0) = 0$ , we say *vanishing* occurs (see Definition 2.3 for detail). We say a positive number  $c^*$  is a *spreading speed* of (1.1) if for any  $h_0 > 0$  and  $u_0$  satisfying (1.2) such that the spreading occurs,

$$\lim_{t \rightarrow \infty} \frac{h(t; u_0, h_0)}{t} = c^*$$

and

$$\liminf_{0 \leq x \leq c' t, t \rightarrow \infty} u(t, x; u_0, h_0) > 0 \quad \forall c' < c^*$$

(see Definition 2.3 for detail).

The spreading speed of (1.1) is strongly related to the so called semi-wave solution of the

following free boundary problem associated with (1.1),

$$\begin{cases} u_t = u_{xx} + uf(t, x, u), & -\infty < x < h(t) \\ u(t, h(t)) = 0 \\ h'(t) = -\mu u_x(t, h(t)). \end{cases} \quad (1.6)$$

If  $(u(t, x), h(t))$  is an entire positive solution of (1.6) with  $\liminf_{x \rightarrow \infty} u(t, h(t) - x) > 0$ , it is called a *semi-wave solution* of (1.6).

In the first part of the series of the papers, we studied the spreading and vanishing dichotomy for (1.1). Under proper assumptions (see (H1)-(H5) in Section 2 of part I, [16]), we proved

• *There are  $l^* > 0$  and a unique time almost periodic positive solution  $u^*(t, x)$  of the following fixed boundary problem,*

$$\begin{cases} u_t = u_{xx} + uf(t, x, u), & x > 0 \\ u_x(t, 0) = 0 \end{cases} \quad (1.7)$$

*such that for any given  $h_0 > 0$  and  $u_0$  satisfying (1.2), either*

- (i)  $h_\infty \leq l^*$  and  $u(t, x; u_0, h_0) \rightarrow 0$  as  $t \rightarrow \infty$  or
- (ii)  $h_\infty = \infty$  and  $u(t, x; u_0, h_0) - u^*(t, x) \rightarrow 0$  as  $t \rightarrow \infty$  locally uniformly in  $x \geq 0$  (see [16, Theorems 2.1 and 2.2] or Proposition 2.1 in the case  $f(t, x, u) \equiv f(t, u)$ ).

In this second part of the series of papers, we study the existence of spreading speeds of (1.1) and semi-wave solutions of (1.6) in the case that  $f(t, x, u) \equiv f(t, u)$ , that is, we consider

$$\begin{cases} u_t = u_{xx} + uf(t, u), & t > 0, 0 < x < h(t) \\ h'(t) = -\mu u_x(t, h(t)), & t > 0 \\ u_x(t, 0) = u(t, h(t)) = 0, & t > 0 \\ h(0) = h_0, u(0, x) = u_0(x), & 0 \leq x \leq h_0. \end{cases} \quad (1.8)$$

Note that (1.6) then becomes

$$\begin{cases} u_t = u_{xx} + uf(t, u), & -\infty < x < h(t) \\ u(t, h(t)) = 0 \\ h'(t) = -\mu u_x(t, h(t)). \end{cases} \quad (1.9)$$

To study the existence of spreading speeds of (1.8) and semi-wave solutions of (1.9), we also consider the following fixed boundary problem on half line,

$$\begin{cases} u_t = u_{xx} - \mu u_x(t, 0)u_x(t, x) + uf(t, u), & 0 < x < \infty \\ u(t, 0) = 0. \end{cases} \quad (1.10)$$

Observe that if  $u^*(t, x)$  is an almost periodic positive solution of (1.10) with  $\liminf_{x \rightarrow \infty} u^*(t, x) > 0$ , let  $u^{**}(t, x) = u^*(t, h^{**}(t) - x)$  and  $h^{**}(t) = \mu \int_0^t u_x^*(s, 0) ds$ . Then  $(u^{**}(t, x), h^{**}(t))$  is an almost periodic semi-wave solution of (1.9). Hence a positive entire solution of (1.10) gives rise to a semi-wave solution of (1.9), and vice visa. Among others, we prove

- *There is a unique time almost periodic stable positive solution  $u^*(t, x)$  of (1.10) satisfying that  $\liminf_{x \rightarrow \infty} u^*(t, x) > 0$  and  $u_x^*(t, 0) > 0$  (hence there is a time almost periodic semi-wave solution of (1.9)) (see Theorem 2.1 for the detail).*
- *$c^* = \mu \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u_x^*(s, 0) ds$  is the spreading speed of (1.8) (see Theorem 2.2 for the detail).*

We remark that, when  $f(t, u)$  is periodic in  $t$  with period  $T$ , the authors of [7] used the following approach to prove the existence of time periodic positive solution of (1.10). First, for any given nonnegative time  $T$ -periodic function  $k(t)$  ( $k(t+T) = k(t)$ ), they prove that there is a unique time  $T$ -periodic positive solution  $U^*(t, x; k(\cdot))$  of the following equation,

$$\begin{cases} u_t = u_{xx} - k(t)u_x + uf(t, u), & 0 < x < \infty \\ u(t, 0) = 0, u(t, x) = u(t+T, x). \end{cases}$$

Then by applying the Schauder fixed point theorem, they prove that there is a nonnegative time  $T$ -periodic function  $k^*(t)$  such that

$$k^*(t) = \mu U_x^*(t, 0; k^*(\cdot)).$$

It then follows that  $u^*(t, x) = U^*(t, x; k^*(\cdot))$  is a time  $T$ -periodic positive solution of (1.10). The application of this approach to the time periodic case is nontrivial. It is difficult to apply this approach to the case that  $f(t, u)$  is almost periodic in  $t$ . We therefore prove the existence of time almost periodic positive solution  $u^*(t, x)$  directly. The proof is certainly also nontrivial and can be applied to the time periodic case as well as more general time dependent cases.

We also remark that similar results to the above hold for the following double fronts free boundary problem:

$$\begin{cases} u_t = u_{xx} + uf(t, u) & t > 0, g(t) < x < h(t) \\ u(t, g(t)) = 0, g'(t) = -\mu u_x(t, g(t)) & t > 0 \\ u(t, h(t)) = 0, h'(t) = -\mu u_x(t, h(t)) & t > 0, \end{cases} \quad (1.11)$$

where both  $x = g(t)$  and  $x = h(t)$  are to be determined.

The rest of this paper is organized as follows. In Section 2, we introduce the definitions and standing assumptions and state the main results of the paper. We present preliminary materials in Section 3 for the use in later sections. Section 4 is devoted to the investigation of time almost periodic KPP equations (1.9) and (1.10) and Theorem 2.1 is proved in this section. We show the existence and provide a characterization of the spreading speed of (1.8) and prove Theorem 2.2 in Section 5. The paper is ended up with some remarks on the spreading speeds and semi-wave solutions of (1.11) in Section 6.

## 2 Definitions, Assumptions, and Main Results

In this section, we introduce the definitions and standing assumptions, and state the main results.

## 2.1 Definitions and assumptions

In this subsection, we introduce the definitions and standing assumptions. We first recall the definition of almost periodic functions, next recall the definition of principal Lyapunov exponents for some linear parabolic equations, then state the standing assumptions, and finally introduce the definition of spreading and vanishing for (1.8).

**Definition 2.1** (Almost periodic function). (1) A continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is called almost periodic if for any  $\epsilon > 0$ , the set

$$T(\epsilon) = \{\tau \in \mathbb{R} \mid |g(t + \tau) - g(t)| < \epsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense in  $\mathbb{R}$ .

(2) Let  $g(t, x, u)$  be a continuous function of  $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ .  $g$  is said to be almost periodic in  $t$  uniformly with respect to  $x \in \mathbb{R}^m$  and  $u$  in bounded sets if  $g$  is uniformly continuous in  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^m$ , and  $u$  in bounded sets and for each  $x \in \mathbb{R}^m$  and  $u \in \mathbb{R}^n$ ,  $g(t, x, u)$  is almost periodic in  $t$ .

(3) For a given almost periodic function  $g(t, x, u)$ , the hull  $H(g)$  is defined by

$$H(g) = \{\tilde{g}(\cdot, \cdot, \cdot) \mid \exists t_n \rightarrow \infty \text{ such that } g(t + t_n, x, u) \rightarrow \tilde{g}(t, x, u) \text{ uniformly in } t \in \mathbb{R}, \\ (x, u) \text{ in bounded sets}\}.$$

**Remark 2.1.** (1) Let  $g(t, x, u)$  be a continuous function of  $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ .  $g$  is almost periodic in  $t$  uniformly with respect to  $x \in \mathbb{R}^m$  and  $u$  in bounded sets if and only if  $g$  is uniformly continuous in  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^m$ , and  $u$  in bounded sets and for any sequences  $\{\alpha'_n\}$ ,  $\{\beta'_n\} \subset \mathbb{R}$ , there are subsequences  $\{\alpha_n\} \subset \{\alpha'_n\}$ ,  $\{\beta_n\} \subset \{\beta'_n\}$  such that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} g(t + \alpha_n + \beta_m, x, u) = \lim_{n \rightarrow \infty} g(t + \alpha_n + \beta_n, x, u)$$

for each  $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$  (see [11, Theorems 1.17 and 2.10]).

(2) We may write  $g(\cdot + t, \cdot, \cdot)$  as  $g \cdot t(\cdot, \cdot, \cdot)$ .

For a given positive constant  $l > 0$  and a given  $C^1$  function  $a(t, x)$  with both  $a(t, x)$  and  $a_t(t, x)$  being almost periodic in  $t$  uniformly in  $x$  in bounded sets, consider

$$\begin{cases} v_t = v_{xx} + a(t, x)v, & 0 < x < l \\ v_x(t, 0) = v(t, l) = 0. \end{cases} \quad (2.1)$$

Let

$$Y(l) = \{u \in C([0, l]) \mid u(l) = 0\}$$

with the norm  $\|u\| = \max_{x \in [0, l]} |u(x)|$  for  $u \in Y(l)$ . Let  $A = \Delta$  acting on  $Y(l)$  with  $\mathcal{D}(A) = \{u \in C^2([0, l]) \cap Y(l) \mid u_x(0) = 0\}$ . Note that  $A$  is a sectorial operator. Let  $0 < \alpha < 1$  be such that  $\mathcal{D}(A^\alpha) \subset C^1([0, l])$ . Fix such an  $\alpha$ . Let

$$X(l) = \mathcal{D}(A^\alpha). \quad (2.2)$$

Then  $X(l)$  is strongly ordered Banach spaces with positive cone

$$X^+(l) = \{u \in X(l) \mid u(x) \geq 0\}.$$

Let

$$X^{++}(l) = \text{Int}(X^+(l)).$$

If no confusion occurs, we may write  $X(l)$  as  $X$ .

By semigroup theory (see [26]), for any  $v_0 \in X(l)$ , (2.1) has a unique solution  $v(t, \cdot; v_0, a)$  with  $v(0, \cdot; v_0, a) = v_0(\cdot)$ .

For given constants  $l > 0$ ,  $\gamma \geq 0$ , and a given  $C^1$  function  $a(t, x)$  with both  $a(t, x)$  and  $a_t(t, x)$  being almost periodic function in  $t$  uniformly in  $x$  in bounded sets, consider also

$$\begin{cases} v_t = v_{xx} - \gamma v_x + a(t, x)v, & 0 < x < l \\ v(t, 0) = v(t, l) = 0. \end{cases} \quad (2.3)$$

Let

$$\tilde{Y}(l) = \{u \in C([0, l]) \mid u(0) = u(l) = 0\}.$$

Let  $A = \Delta$  acting on  $\tilde{Y}(l)$  with  $\mathcal{D}(A) = \{u \in C^2([0, l]) \cap \tilde{Y}(l)\}$ . Note that  $A$  is a sectorial operator. Let  $0 < \alpha < 1$  be such that  $\mathcal{D}(A^\alpha) \subset C^1([0, l])$ . Fix such an  $\alpha$ . Let

$$\tilde{X}(l) = \mathcal{D}(A^\alpha). \quad (2.4)$$

Then, for any  $v_0 \in \tilde{X}(l)$ , (2.3) has a unique solution  $\tilde{v}(t, \cdot; v_0, a)$  with  $\tilde{v}(0, \cdot; v_0, a) = v_0(\cdot)$ .

**Definition 2.2** (Principal Lyapunov exponent). (1) Let  $V(t, a)v_0 = v(t, \cdot; v_0, a)$  for  $v_0(\cdot) \in X(l)$  and

$$\lambda(a, l) = \limsup_{t \rightarrow \infty} \frac{\ln \|V(t, a)\|_{X(l)}}{t}.$$

$\lambda(a, l)$  is called the principal Lyapunov exponent of (2.1).

(2) Let

$$\tilde{\lambda}(a, \gamma, l) = \limsup_{t \rightarrow \infty} \frac{\ln \|\tilde{V}(t, a)\|_{\tilde{X}(l)}}{t}$$

where  $\tilde{V}(t, a)v_0 = \tilde{v}(t, \cdot; v_0, a)$  for  $v_0 \in \tilde{X}(l)$ .  $\tilde{\lambda}(a, \gamma, l)$  is called the principal Lyapunov exponent of (2.3).

Let (H1)-(H3) be the following standing assumptions.

**(H1)**  $f(t, u)$  is  $C^1$  in  $(t, u) \in \mathbb{R}^2$ ,  $Df = (f_t, f_u)$  is bounded in  $t \in \mathbb{R}$  and in  $u$  in bounded sets, and  $f$  is monostable in  $u$  in the sense that there are  $M > 0$  such that

$$\sup_{t \in \mathbb{R}, u \geq M} f(t, u) < 0$$

and

$$\sup_{t \in \mathbb{R}, u \geq 0} f_u(t, u) < 0.$$

**(H2)**  $f(t, u)$  and  $Df(t, u) = (f_t(t, u), f_u(t, u))$  are almost periodic in  $t$  uniformly with respect to  $u$  in bounded sets.

**(H3)**  $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s, 0) ds > 0$ .

Assume (H1) and (H2). We remark that (H3) implies that there are  $L^* \geq l^* > 0$  such that  $\lambda(a(\cdot), l) > 0$  for  $l > l^*$  and  $\tilde{\lambda}(a(\cdot), 0, l) > 0$  for  $l > L^*$ , where  $a(t) = f(t, 0)$  (see Lemma 3.2 and Remark 3.1 for the reasonings).

Consider (1.8). Throughout this paper, we assume (H1)-(H3). For any given  $h_0 > 0$  and  $u_0$  satisfying (1.2), (1.8) has a unique solution  $(u(t, x; u_0, h_0), h(t; u_0, h_0))$  with  $u(0, x; u_0, h_0) = u_0(x)$  and  $h(0; u_0, h_0) = h_0$  (see [7]). By comparison principle for parabolic equations,  $u(t, x; u_0, h_0)$  exists for all  $t > 0$  and  $u_x(t, h(t; u_0, h_0); u_0, h_0) < 0$  for  $t > 0$ . Hence  $h(t; u_0, h_0)$  is monotonically increasing, and therefore there exists  $h_\infty \in (0, +\infty]$  such that  $\lim_{t \rightarrow +\infty} h(t; u_0, h_0) = h_\infty$ .

**Definition 2.3** (Spreading-vanishing and spreading speed). *Consider (1.8).*

- (1) For given  $h_0 > 0$  and  $u_0$  satisfying (1.2), let  $h_\infty = \lim_{t \rightarrow \infty} h(t; u_0, h_0)$ . It is said that the vanishing occurs if  $h_\infty < \infty$  and  $\lim_{t \rightarrow \infty} \|u(t, \cdot; u_0, h_0)\|_{C([0, h(t)])} = 0$ . It is said that the spreading occurs if  $h_\infty = \infty$  and  $\liminf_{t \rightarrow \infty} u(t, x; u_0, h_0) > 0$  locally uniformly in  $x \in [0, \infty)$ .
- (2) A real number  $c^* > 0$  is called the spreading speed of (1.8) if for any  $h_0 > 0$  and  $u_0$  satisfying (1.2) such that the spreading occurs, there holds

$$\lim_{t \rightarrow \infty} \frac{h(t; u_0, h_0)}{t} = c^*$$

and

$$\liminf_{0 \leq x \leq c' t, t \rightarrow \infty} u(t, x; u_0, h_0) > 0, \quad \forall c' < c^*.$$

Assume (H1)-(H3). It is known that there is a unique time almost periodic positive solution  $V^*(t)$  of the following ODE (see Lemma 3.3),

$$u_t = uf(t, u).$$

**Definition 2.4.** An entire positive solution  $(u(t, x), h(t))$  of (1.9) is called an almost periodic semi-wave solution if  $u(t, h(t) - x)$  is almost periodic in  $t$  uniformly with respect to  $x \geq 0$  and  $h'(t)$  is almost periodic in  $t$ , and  $\lim_{x \rightarrow \infty} u(t, h(t) - x) = V^*(t)$  uniformly in  $t \in \mathbb{R}$ .

**Remark 2.2.** If  $\tilde{u}^{**}(t, x)$  is an almost periodic positive solution of (1.10) uniformly with respect to  $x \geq 0$  and  $\lim_{x \rightarrow \infty} \tilde{u}^{**}(t, x) = V^*(t)$  uniformly in  $t$ , then  $(u^{**}(t, x), h^{**}(t))$  is an almost periodic semi-wave solution of (1.9), where  $u^{**}(t, x) = \tilde{u}^{**}(t, h^{**}(t) - x)$  and  $h^{**}(t) = \mu \int_0^t \tilde{u}_x^{**}(s, 0) ds$ .



## 2.2 Main results

In this subsection, we state the main results of this paper. To do so, we first recall the main results obtained in the part I of the series.

**Proposition 2.1.** *Assume (H1)-(H3). For any given  $h_0 > 0$  and  $u_0$  satisfying (1.2), let  $(u(t, x; u_0, h_0), h(t; u_0, h_0))$  be the solution of (1.8) with  $(u(0, x; u_0, h_0), h(0; u_0, h_0)) = (u_0(x), h_0)$ . Then either*

- (i)  $h_\infty \leq l^*$  and  $u(t, x; u_0, h_0) \rightarrow 0$  as  $t \rightarrow \infty$  or
- (ii)  $h_\infty = \infty$  and  $u(t, x; u_0, h_0) - V^*(t) \rightarrow 0$  as  $t \rightarrow \infty$  locally uniformly in  $x \geq 0$ .

*Proof.* See [16, Theorem 2.2]. □

The main results of this paper are stated in the following two theorems.

**Theorem 2.1** (Almost periodic semi-waves). *Assume (H1)-(H3).*

- (1) *There is a time almost periodic solution  $\tilde{u}^{**}(t, x)$  of (1.10) with  $\lim_{x \rightarrow \infty} \tilde{u}^{**}(t, x) = V^*(t)$  uniformly in  $t \in \mathbb{R}$  and hence there is a time almost periodic positive semi-wave solution  $(u^{**}(t, x), h^{**}(t))$  of (1.9) with  $h^{**}(0) = 0$ .*
- (2) *If  $\tilde{u}_1^{**}(t, x)$  and  $\tilde{u}_2^{**}(t, x)$  are two almost periodic positive solutions of (1.10) satisfying that  $\lim_{x \rightarrow \infty} \tilde{u}_i^{**}(t, x) = V^*(t)$  uniformly in  $t \in \mathbb{R}$  ( $i = 1, 2$ ), then  $\tilde{u}_1^{**}(t, x) \equiv \tilde{u}_2^{**}(t, x)$ .*
- (3) *For any bounded positive solution  $\tilde{u}(t, x)$  of (1.10) with  $\liminf_{x \rightarrow \infty} \inf_{t \geq 0} \tilde{u}(t, x) > 0$ ,*

$$\lim_{t \rightarrow \infty} [\tilde{u}^{**}(t, x) - \tilde{u}(t, x)] = 0$$

*uniformly in  $x \geq 0$ .*

**Theorem 2.2** (Spreading speed and semi-wave). *Assume (H1)-(H3) and  $f(t, x, u) \equiv f(t, u)$ . Let  $(u^{**}(t, x), h^{**}(t))$  be as in Theorem 2.1 (1), and*

$$c^* = \lim_{t \rightarrow \infty} \frac{h^{**}(t)}{t}.$$

*Then  $c^*$  is the spreading speed of (1.8), that is, for any given  $h_0 > 0$  and  $u_0$  satisfying (1.2), if  $h_\infty = \lim_{t \rightarrow \infty} h(t; u_0, h_0) = \infty$ , then  $\lim_{t \rightarrow \infty} \frac{h(t; u_0, h_0)}{t} = c^*$  and*

$$\lim_{t \rightarrow \infty} \max_{x \leq (c^* - \epsilon)t} |u(t, x; u_0, h_0) - V^*(t)| = 0$$

*for every small  $\epsilon > 0$ .*

### 3 Preliminary

In this section, we present some preliminary results to be applied in later sections, including basic properties for principal Lyapunov exponents (see subsection 3.1), the asymptotic dynamics of some diffusive KPP equations with time almost periodic dependence in fixed environments (see subsection 3.2), and comparison principles for free boundary problems (see subsection 3.3).

#### 3.1 Principal Lyapunov exponents

Consider (2.1). Let  $X = X(l)$ , where  $X(l)$  is as in (2.2). We denote by  $\|\cdot\|$  the norm in  $X$  or in  $\mathcal{L}(X, X)$ . Recall that for any  $v_0 \in X$ , (2.1) has a unique solution  $v(t, \cdot; v_0, a)$  and

$$\lambda(a, l) = \limsup_{t \rightarrow \infty} \frac{\ln \|V(t, a)\|}{t}$$

where  $V(t, a)v_0 = v(t, \cdot; v_0, a)$ . For any  $b \in H(a)$ , consider also

$$\begin{cases} v_t = v_{xx} + b(t, x)v, & 0 < x < l \\ v_x(t, 0) = v(t, l) = 0, \end{cases} \quad (3.1)$$

For any  $v_0 \in X$ , (3.1) has also a unique solution  $v(t, \cdot; v_0, b)$  with  $v(0, \cdot; v_0, b) = v_0$ .

**Lemma 3.1.** *There is  $\phi_l : H(a) \rightarrow X^{++}$  satisfying the following properties.*

- (i)  $\|\phi_l(b)\| = 1$  for any  $b \in H(a)$  and  $\phi_l : H(a) \rightarrow X^{++}$  is continuous.
- (ii)  $v(t, \cdot; \phi_l(b), b) = \|v(t, \cdot; \phi_l(b), b)\| \phi_l(b(\cdot + t, \cdot))$ .
- (iii)  $\lim_{t \rightarrow \infty} \frac{\ln \|v(t, \cdot; \phi_l(b), b)\|}{t} = \lambda(a, l)$  uniformly in  $b \in H(a)$ .

*Proof.* It follows from [19] (see also [20, 31]). □

**Lemma 3.2.** *Suppose that  $a(t, x) \equiv a(t)$ . Then*

$$\lambda(a, l) = \hat{a} + \lambda_0(l),$$

where  $\hat{a} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t a(s) ds$  and  $\lambda_0(l)$  is the principal eigenvalue of

$$\begin{cases} u_{xx} = \lambda u, & 0 < x < l \\ u_x(0) = u(l) = 0. \end{cases} \quad (3.2)$$

*Proof.* Let  $\tilde{v}(t, x) = v(t, x)e^{-\int_0^t a(s) ds}$ . Then (2.1) becomes

$$\begin{cases} \tilde{v}_t = \tilde{v}_{xx}, & 0 < x < l \\ \tilde{v}_x(t, 0) = \tilde{v}(t, l) = 0. \end{cases}$$

It then follows that  $\lambda(a, l) = \hat{a} + \lambda(0, l)$ . It is clear that  $\lambda(0, l) = \lambda_0(l)$ . The lemma then follows. □

**Remark 3.1.** (1) *Principal Lyapunov exponent theory for (2.1) also holds for (2.3).*

(2) *When  $a(t, x) \equiv a(t)$ ,  $\tilde{\lambda}(a, \gamma, l) = \hat{a} + \tilde{\lambda}(0, \gamma, l)$ . Note that  $\lambda(0, l) = -\frac{\pi^2}{4l^2}$  and  $\tilde{\lambda}(0, \gamma, l) = -\left(\frac{\gamma^2}{4} + \frac{\pi^2}{l^2}\right)$ . Hence  $\lambda(a, l) > 0$  and  $\tilde{\lambda}(a, 0, l) > 0$  for  $l \gg 1$ .*

### 3.2 Asymptotic dynamics of diffusive KPP equations with time almost periodic dependence in fixed domains

In this subsection, we consider the asymptotic dynamics of the following KPP equations,

$$u_t = uf(t, u), \quad (3.3)$$

$$\begin{cases} u_t = u_{xx} + uf(t, u), & x > 0 \\ u_x(t, 0) = 0, \end{cases} \quad (3.4)$$

and

$$\begin{cases} u_t = u_{xx} - \epsilon \mu u_x + uf(t, u), & x > 0 \\ u(t, 0) = 0. \end{cases} \quad (3.5)$$

Throughout this subsection, we assume (H1) and (H2). Let

$$H(f) = \text{cl}\{f(\cdot + \tau, \cdot) \mid \tau \in \mathbb{R}\}$$

where the closure is taken in the open compact topology. Observe that for any  $g \in H(f)$ ,  $g$  also satisfies (H1) and (H2).

First of all, consider (3.3) and

$$u_t = ug(t, u) \quad (3.6)$$

for any  $g \in H(f)$ . By fundamental theory for ordinary differential equations, for any  $u_0 \in \mathbb{R}$  and  $g \in H(f)$ , (3.6) has a unique (local) solution  $u(t; u_0, g)$  with  $u(0; u_0, g) = u_0$ . By (H1), for any  $u_0 \geq 0$ ,  $u(t; u_0, g) \geq 0$  and  $u(t; u_0, g)$  exists for all  $t \geq 0$ .

**Lemma 3.3.** *For any  $g \in H(f)$ , there is a unique stable almost periodic positive solution  $u_g(t)$  of (3.6).*

*Proof.* It follows from [30, Theorem 4.1]. □

Next, consider (3.4) and

$$\begin{cases} u_t = u_{xx} + ug(t, u), & x > 0 \\ u_x(t, 0) = 0. \end{cases} \quad (3.7)$$

for any  $g \in H(f)$ .

Let

$$\tilde{X}_0 = C_{\text{unif}}^b([0, \infty))$$

with the norm  $\|u\| = \sup_{x \in [0, \infty)} |u(x)|$  for  $u \in \tilde{X}_0$ . The operator  $A = \Delta$  with  $\mathcal{D}(A) = \{u \in \tilde{X}_0 \mid u'(\cdot), u''(\cdot) \in \tilde{X}_0, u'(0) = 0\}$  is a sectorial operator. Let

$$\tilde{X} = \text{a fractional power space of } A \text{ such that for any } u \in \tilde{X}, u'(\cdot) \in C_{\text{unif}}^b([0, \infty)). \quad (3.8)$$

Let

$$\tilde{X}^+ = \{u \in \tilde{X} \mid u(x) \geq 0 \text{ for } x \in \mathbb{R}^+\}$$

and

$$\tilde{X}^{++} = \{u \in \tilde{X}^+ \mid \inf_{x \geq 0} u(x) > 0\}.$$

By general semigroup theory, for any  $u_0 \in \tilde{X}$  and any  $g \in H(f)$ , there is a unique (local) solution  $u(t, \cdot; u_0, g)$  with  $u(0, x; u_0, g) = u_0(x)$ . By comparison principle for parabolic equations, for any  $u_0 \in \tilde{X}^+$ ,  $u(t, \cdot; u_0, g) \in \tilde{X}^+$  and  $u(t, \cdot; u_0, g)$  exists for all  $t \geq 0$ . If  $u_0 \in \tilde{X}^{++}$ , then  $u(t, \cdot; u_0, g) \in \tilde{X}^{++}$  for all  $t \geq 0$ .

**Remark 3.2.** For any  $g \in H(f)$ ,  $u(t, x) = u_g(t)$  is an almost periodic solution of (3.7). Moreover, for any  $u_0 \in \tilde{X}^{++}$ ,

$$u(t, x; u_0, g) - u_g(t) \rightarrow 0$$

as  $t \rightarrow \infty$  uniformly in  $x \geq 0$ .

Consider now (3.5) and

$$\begin{cases} u_t = u_{xx} - \epsilon \mu u_x + u g(t, u), & x > 0 \\ u(t, 0) = 0 \end{cases} \quad (3.9)$$

for any  $g \in H(f)$ .

Let

$$\hat{X}_0 = \{u \in C_{\text{unif}}^b([0, \infty)) \mid u(0) = 0\}$$

with the norm  $\|u\| = \sup_{x \in [0, \infty)} |u(x)|$  for  $u \in \hat{X}_0$ . The operator  $A = \Delta$  with  $\mathcal{D}(A) = \{u \in \hat{X}_0 \mid u'(\cdot), u''(\cdot) \in C_{\text{unif}}^b([0, \infty))\}$  is a sectorial operator. Let

$$\hat{X} = \text{a fractional power space of } A \text{ such that for any } u \in \hat{X}, u'(\cdot) \in C_{\text{unif}}^b([0, \infty)). \quad (3.10)$$

Let

$$\hat{X}^+ = \{u \in \hat{X} \mid u(x) \geq 0 \text{ for } x \in \mathbb{R}^+\}$$

and

$$\hat{X}^{++} = \{u \in \hat{X}^+ \mid \inf_{x \geq \epsilon} u(x) > 0 \text{ for any } \epsilon > 0 \text{ and } u'(0) > 0\}.$$

By general semigroup theory, for any  $u_0 \in \hat{X}$  and any  $g \in H(f)$ , there is a unique (local) solution  $u(t, \cdot; u_0, g)$  with  $u(0, x; u_0, g) = u_0(x)$ . By comparison principle for parabolic equations, for any  $u_0 \in \hat{X}^+$ ,  $u(t, \cdot; u_0, g) \in \hat{X}^+$  and  $u(t, \cdot; u_0, g)$  exists for all  $t \geq 0$ . If  $u_0 \in \hat{X}^{++}$ , then  $u(t, \cdot; u_0, g) \in \hat{X}^{++}$  for all  $t \geq 0$ .

By Remark 3.1, there are  $\tilde{l}^* > 0$  and  $\epsilon^* > 0$  such that  $\tilde{\lambda}(a, \epsilon \mu, l) > 0$  for  $l > \tilde{l}^*$  and  $0 < \epsilon < \epsilon^*$ .

**Lemma 3.4.** Let  $\epsilon > 0$  be given such that  $\tilde{\lambda}(a, \epsilon\mu, l) > 0$  for  $l \gg 1$ . For given  $g \in H(f)$ , Consider

$$\begin{cases} u_t = u_{xx} - \epsilon\mu u_x + ug(t, u), & x > 0 \\ u(t, 0) = 0. \end{cases} \quad (3.11)$$

For any  $u_0 \in \hat{X}^{++}$ ,  $\inf_{t \geq 0, g \in H(f)} \partial_x u_\epsilon(t, 0; u_0, g) > 0$ .

*Proof.* First of all, we consider the following problem

$$\begin{cases} u_t = u_{xx} - \epsilon\mu u_x + ug(t, u), & 0 < x < l \\ u(t, 0) = u(t, l) = 0 \end{cases} \quad (3.12)$$

Since  $\tilde{\lambda}(a, \epsilon\mu, l) > 0$ , there is a unique time almost periodic positive stable solution  $u_{\epsilon, g}^l(t, x)$  of (3.12). Moreover, for any  $\tilde{u}_0 \in C([0, l])$  with  $\tilde{u}_0(0) = \tilde{u}_0(l) = 0$  and  $\tilde{u}_0(x) > 0$  for  $x \in (0, l)$ ,

$$\lim_{t \rightarrow \infty} |u_\epsilon^l(t, x; \tilde{u}_0, g) - u_{\epsilon, g}^l(t, x)| = 0$$

uniformly in  $x \in [0, l]$  and  $g \in H(f)$ , and

$$\lim_{t \rightarrow \infty} |\partial_x u_\epsilon^l(t, 0; \tilde{u}_0, g) - \partial_x u_{\epsilon, g}^l(t, 0)| = 0$$

uniformly in  $g \in H(f)$ , where  $u_\epsilon^l(t, x; \tilde{u}_0, g)$  is the solution of (3.12) with  $u_\epsilon^l(0, x; \tilde{u}_0, g) = \tilde{u}_0(x)$ .

Now for any  $u_0 \in \hat{X}^{++}$ , choose  $\tilde{u}_0 \in C([0, l])$  such that  $\tilde{u}_0(0) = \tilde{u}_0(l) = 0$ ,  $\tilde{u}_0(x) > 0$  for  $x \in (0, l)$ ,  $\partial_x u_0(0) > 0$ , and

$$\tilde{u}_0(x) \leq u_0(x) \quad \text{for } 0 \leq x \leq l.$$

Then by comparison principle for parabolic equations, we have

$$u_\epsilon(t, x; u_0, g) \geq u_\epsilon^l(t, x; \tilde{u}_0, g) \quad \text{for } 0 \leq x \leq l.$$

This implies that

$$\partial_x u_\epsilon(t, 0; u_0, g) \geq \partial_x u_\epsilon^l(t, 0; \tilde{u}_0, g)$$

and then

$$\inf_{t \geq 0, g \in H(f)} \partial_x u_\epsilon(t, 0; u_0, g) > 0.$$

This proves the lemma. □

### 3.3 Comparison principal for free boundary problems

In order for later application, we present some comparison principles for free boundary problems in this subsection.

**Proposition 3.1.** Let  $f(t, u)$  be a function satisfying (H1) and (H2). Suppose that  $T \in (0, \infty)$ ,  $\bar{h} \in C^1([0, T])$ ,  $\bar{u} \in C(\bar{D}_T^*) \cap C^{1,2}(D_T^*)$  with  $D_T^* = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, 0 < x < \bar{h}(t)\}$ , and

$$\begin{cases} \bar{u}_t \geq \bar{u}_{xx} + \bar{u}f(t, \bar{u}), & t > 0, 0 < x < \bar{h}(t) \\ \bar{h}'(t) \geq -\mu \bar{u}_x(t, \bar{h}(t)), & t > 0, \\ \bar{u}_x(t, 0) \leq 0, u(t, \bar{h}(t)) = 0, & t > 0 \end{cases}$$

If  $h_0 \leq \bar{h}(0)$  and  $u_0(x) \leq \bar{u}(0, x)$  in  $[0, h_0]$ , then the solution  $(u, h)$  of the free boundary problem (1.8) satisfies

$$h(t) \leq \bar{h}(t) \text{ for all } t \in (0, T], \quad u(t, x) \leq \bar{u}(t, x) \text{ for } t \in (0, T] \text{ and } x \in (0, h(t)).$$

*Proof.* The proof of this Proposition is similar to that of Lemma 3.5 in [9] and Lemma 2.6 in [6].  $\square$

**Remark 3.3.** The pair  $(\bar{u}, \bar{h})$  in Proposition 3.1 is called an upper solution of the free boundary problem. We can define a lower solution by reversing all the inequalities in the obvious places.

**Proposition 3.2.** Let  $f(t, u)$  be a function satisfying (H1) and (H2). Suppose that  $T \in (0, \infty)$ ,  $\bar{h} \in C^1([0, T])$ ,  $\bar{u} \in C^{1,2}(D_T^*)$  with  $D_T^* = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, -\infty < x < \bar{h}(t)\}$ , and

$$\begin{cases} \bar{u}_t \geq \bar{u}_{xx} + \bar{u}f(t, \bar{u}), & t > 0, -\infty < x < \bar{h}(t) \\ \bar{h}'(t) \geq -\mu \bar{u}_x(t, \bar{h}(t)), & t > 0, \\ u(t, \bar{h}(t)) = 0, & t > 0 \end{cases}$$

If  $h_0 \leq \bar{h}(0)$  and  $u_0(x) \leq \bar{u}(0, x)$  in  $(-\infty, h_0]$ , then the solution  $(u, h)$  of the free boundary problem

$$\begin{cases} u_t = u_{xx} + uf(t, u), & t > 0, -\infty < x < h(t) \\ h'(t) = -\mu u_x(t, h(t)), & t > 0 \\ u(t, h(t)) = 0, & t > 0 \\ h(0) = h_0, u(0, x) = u_0(x), & -\infty < x \leq h_0 \end{cases}$$

satisfies

$$h(t) \leq \bar{h}(t) \text{ for all } t \in (0, T], \quad u(t, x) \leq \bar{u}(t, x) \text{ for } t \in (0, T] \text{ and } x \in (-\infty, h(t)).$$

*Proof.* The proof of this Proposition is similar to Proposition 3.1.  $\square$

**Proposition 3.3.** For any given  $h_0 > 0$  and  $u_0$  satisfying (1.2),  $(u(t, x; u_0, h_0), h(t; u_0, h_0))$  exists for all  $t \geq 0$ .

*Proof.* The proof is similar to that of Theorem 4.3 in [6].  $\square$

**Remark 3.4.** From the uniqueness of the solution to (1.8) and some standard compactness argument, we can obtain that the unique solution  $(u, h)$  depends continuously on  $u_0$  and the parameters appearing in (1.8).

We will need some simple variants of Proposition 3.1 and Remark 3.3, whose proofs are similar to the original ones and therefore omitted.

**Lemma 3.5.** *Let  $f(t, u)$  be a function satisfying (H1) and (H2). Suppose that  $T \in (0, \infty)$ ,  $\bar{h} \in C^1([0, T])$ ,  $\bar{u} \in C^{1,2}(D_T^*)$  with  $D_T^* = \{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq x \leq \bar{h}(t)\}$ , and*

$$\begin{cases} \bar{u}_t \geq \bar{u}_{xx} + \bar{u}f(t, \bar{u}), & t \in (0, T], 0 < x < \bar{h}(t), \\ \bar{u}(t, \bar{h}(t)) = 0, \bar{h}'(t) \geq -\mu\bar{u}_x(t, \bar{h}(t)), & t \in (0, T], \\ \bar{u}(t, 0) \geq l(t), & t \in (0, T]. \end{cases}$$

*If  $h \in C^1([0, T])$  and  $u \in C^{1,2}(D_T)$  with  $D_T = \{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq x \leq h(t)\}$  satisfy*

$$0 < h(0) \leq \bar{h}(0), 0 < u(0, x) \leq \bar{u}(0, x) \text{ for } 0 \leq x \leq h(0),$$

*and*

$$\begin{cases} u_t = u_{xx} + uf(t, u), & t \in (0, T], 0 < x < h(t), \\ u(t, h(t)) = 0, h'(t) = -\mu u_x(t, h(t)), & t \in (0, T], \\ u(t, 0) = l(t), & t \in (0, T]. \end{cases} \quad (3.13)$$

*then*

$$h(t) \leq \bar{h}(t) \text{ for } t \in (0, T], u(t, x) \leq \bar{u}(t, x) \text{ for } (t, x) \in (0, T] \times (0, h(t)).$$

Similarly, we have the following analogue of Lemma 3.5.

**Lemma 3.6.** *Let  $f(t, u)$  be as in Lemma 3.5. Suppose that  $T \in (0, \infty)$ ,  $\underline{h} \in C^1([0, T])$ ,  $\underline{u} \in C^{1,2}(D_T^+)$  with  $D_T^+ = \{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq x \leq \underline{h}(t)\}$ , and*

$$\begin{cases} \underline{u}_t \leq \underline{u}_{xx} + \underline{u}f(t, \underline{u}), & t \in (0, T], 0 < x < \underline{h}(t), \\ \underline{u}(t, \underline{h}(t)) = 0, \underline{h}'(t) \leq -\mu\underline{u}_x(t, \underline{h}(t)), & t \in (0, T], \\ \underline{u}(t, 0) \leq l(t), & t \in (0, T]. \end{cases}$$

*If  $h \in C^1([0, T])$  and  $u \in C^{1,2}(D_T^+)$  satisfy (3.13) and*

$$h(0) \geq \underline{h}(0), u(0, x) \geq \underline{u}(0, x) \geq 0, \text{ for } 0 \leq x \leq \underline{h}(0),$$

*then*

$$h(t) \geq \underline{h}(t) \text{ for } t \in (0, T], u(t, x) \geq \underline{u}(t, x) \text{ for } (t, x) \in (0, T] \times (0, \underline{h}(t)).$$

## 4 Basic Properties of Diffusive KPP Equations in Unbounded Domains

In this section, we present some basic properties of (1.9) and (1.10). Throughout this subsection, we assume (H1) and (H2). Let

$$H(f) = \text{cl}\{f(\cdot + \tau, \cdot) \mid \tau \in \mathbb{R}\},$$

where the closure is taken in the open compact topology. Observe that for any  $g \in H(f)$ ,  $g$  also satisfies (H1) and (H2).

Consider (1.10) and

$$\begin{cases} u_t = u_{xx} - \mu u_x(t, 0)u_x(t, x) + ug(t, u), & 0 < x < \infty \\ u(t, 0) = 0 \end{cases} \quad (4.1)$$

for any  $g \in H(f)$ .

By general semigroup theory, for any  $u_0 \in \hat{X}$ , (4.1) has a unique solution  $u(t, \cdot; u_0, g)$  with  $u(0, \cdot; u_0, g) = u_0$ . By (H1) and comparison principle for parabolic equations, we have that for any  $u_0 \in \hat{X}^+$ ,  $u(t, \cdot; u_0, g)$  exists and  $u(t, \cdot; u_0, g) \in \hat{X}^+$  for all  $t > 0$ . Moreover, there is a constant  $M(u_0) > 0$  such that  $u(t, \cdot; u_0, g) \leq M(u_0)$  and  $|u_x(t, x; u_0, g)| \leq M(u_0)$  for  $t \geq 0$  and  $g \in H(f)$ .

Consider (1.9) and

$$\begin{cases} u_t = u_{xx} + ug(t, u), & -\infty < x < h(t) \\ u(t, h(t)) = 0 \\ h'(t) = -\mu u_x(t, h(t)) \end{cases} \quad (4.2)$$

for any  $g \in H(f)$ .

Note that a solution  $u(t, x)$  of (4.1) gives rise to a solution  $(\tilde{u}(t, x), \tilde{h}(t))$  of (4.2), where  $\tilde{u}(t, x) = u(t, \tilde{h}(t) - x)$  and  $\tilde{h}(t) = \mu \int_0^t u_x(s, 0)ds$ . Conversely, a solution  $(u(t, x), h(t))$  of (4.2) gives rise to a solution  $\tilde{u}(t, x)$  of (4.1), where  $\tilde{u}(t, x) = u(t, h(t) - x)$ . Note also that for given  $h_0 \in \mathbb{R}$  and  $u_0(\cdot)$  satisfying

$$u_0(h_0) = 0, \quad u_0(h_0 - \cdot) \in \hat{X}^+, \quad (4.3)$$

(4.2) has a unique solution  $(u(t, x; u_0, h_0, g), h(t; u_0, h_0, g))$  with  $(u(0, x; u_0, h_0, g), h(0; u_0, h_0, g)) = (u_0(x), h_0)$ .

#### 4.1 Basic properties of diffusive KPP equations in unbounded domains with a free boundary

In this subsection, we present some basic properties of solutions of (1.9) and (4.2).

For given  $g \in H(f)$ , given  $h_{10}, h_{20} \in \mathbb{R}$  and  $u_{10}$  and  $u_{20}$  satisfying (4.3) with  $h_0$  being replaced by  $h_{10}$  and  $h_{20}$ , respectively, assume that  $h(t; u_{10}, h_{10}, g) \leq h(t; u_{20}, h_{20}, g)$  for  $0 \leq t \leq T$ . Then  $w(t, x) := u(t, x; u_{20}, h_{20}, g) - u(t, x; u_{10}, h_{10}, g)$  satisfies

$$w_t = w_{xx} + a(t, x)w, \quad -\infty < x < \eta(t), \quad 0 < t \leq T, \quad (4.4)$$

where  $\eta(t) = h(t; u_{10}, h_{10}, g)$  and  $a(t, x) = 0$  if  $u(t, x; u_{20}, h_{20}, g) = u(t, x; u_{10}, h_{10}, g)$  and

$$a(t, x) = \frac{u(t, x; u_{20}, h_{20}, g)g(t, u(t, x; u_{20}, h_{20}, g)) - u(t, x; u_{10}, h_{10}, g)g(t, u(t, x; u_{10}, h_{10}, g))}{u(t, x; u_{20}, h_{20}, g) - u(t, x; u_{10}, h_{10}, g)}$$

if  $u(t, x; u_{20}, h_{20}, g) \neq u(t, x; u_{10}, h_{10}, g)$ .



**Lemma 4.1.** *Let  $\eta(t)$  be a continuous function for  $t \in (t_1, t_2)$ . If  $w(t, x)$  is a continuous function for  $t \in (t_1, t_2)$  and  $x \in (-\infty, \eta(t))$ , and satisfies*

$$w_t = w_{xx} + a(t, x)w, \quad x \in (-\infty, \eta(t)), \quad t \in (t_1, t_2)$$

*for some bounded continuous function  $a(t, x)$  and  $w(t, \eta(t)) \neq 0$ ,  $w(t, x) \neq 0$  for  $x \ll -1$ , then for each  $t \in (t_1, t_2)$ , the number of zero (denoted by  $Z(t)$ ) of  $w(t, \cdot)$  in  $(-\infty, \eta(t)]$  is finite. Moreover  $Z(t)$  is nonincreasing in  $t$ , and if for some  $s \in (t_1, t_2)$  the function  $w(s, \cdot)$  has a degenerate zero  $x_0 \in (-\infty, \eta(s))$ , then  $Z(s_1) > Z(s_2)$  for all  $s_1, s_2$  satisfying  $t_1 < s_1 < s < s_2 < t_2$ .*

*Proof.* For any  $t_0 \in (t_1, t_2)$ , by the continuity of  $w$  we can find  $\epsilon > 0, \delta > 0$  and  $M < 0$  such that

$$w(t, x) \neq 0 \quad \text{for } t \in I_{t_0} := (t_0 - \delta, t_0 + \delta), \quad x \in \{M\} \cup [\eta(t_0) - \epsilon, \eta(t)]$$

Without loss of generality, we may assume that

$$w(t_0, x) > 0 \quad \text{for } -\infty < x \leq M.$$

Then

$$w(t, M) > 0 \quad \text{for } t \in (t_0 - \delta, t_0 + \delta).$$

By comparison principle for parabolic equations, we have

$$w(t, x) > 0 \quad \text{for } t \in (t_0, t_0 + \delta), \quad -\infty < x \leq M.$$

Let  $Z(t; M, \eta(t_0) - \epsilon)$  be the number of zeros of  $u(t, \cdot)$  in the interval  $[M, \eta(t_0) - \epsilon]$ . We can apply Theorem D in [1] to see that the conclusions for  $Z(t; M, \eta(t_0) - \epsilon)$  hold for  $t \in I_{t_0}$  and hence  $Z(t) = Z(t; M, \eta(t_0) - \epsilon)$  is finite for  $t \in [t_0, t_0 + \delta)$ . This implies that  $Z(t)$  is finite for any  $t \in (t_1, t_2)$ . Moreover,

$$Z(t) \geq Z(t; M, \eta(t_0) - \epsilon) \geq Z(t_0; M, \eta(t_0) - \epsilon) = Z(t_0) \quad \text{for } t \in (t_0 - \delta, t_0),$$

$$Z(t) = Z(t; M, \eta(t_0) - \epsilon) \leq Z(t_0; M, \eta(t_0) - \epsilon) = Z(t_0) \quad \text{for } t \in (t_0, t_0 + \delta),$$

and if  $w(t_0, \cdot)$  has a degenerate zero  $x_0 \in (-\infty, \eta(t_0))$ , then  $Z(s_1) > Z(s_2)$  for all  $s_1, s_2$  satisfying  $t_1 < s_1 < t_0 < s_2 < t_2$ . □

**Lemma 4.2.** *For given  $g \in H(f)$ ,  $h_{10}, h_{20} \in \mathbb{R}$ , and  $u_{10}, u_{20}$  satisfying (4.3) with  $h_0$  being replaced by  $h_{10}$  and  $h_{20}$ , respectively. If  $u'_{20}(x_2) < u'_{10}(x_1)$  for any  $x_1, x_2$  such that  $u_{20}(x_2) = u_{10}(x_1)$ , then*

$$u(s, x + h(s; u_{20}, h_{20}, g); u_{20}, h_{20}, g) \geq u(s, x + h(s; u_{10}, h_{10}, g); u_{10}, h_{10}, g)$$

*for  $x \leq 0$  and  $s \geq 0$ .*

*Proof.* Fix any  $s > 0$ . Let  $\tilde{u}_1(t, x) = u(t, x + h(s; u_{10}, h_{10}, g); u_{10}, h_{10}, g)$  and  $\tilde{u}_2(t, x) = u(t, x + h(s; u_{20}, h_{20}, g); u_{20}, h_{20}, g)$ . Then

$$\tilde{u}_1(t, x) = u(t, x; u_{10}(\cdot + h(s; u_{10}, h_{10}, g)), h_{10} - h(s; u_{10}, h_{10}, g), g)$$

and

$$\tilde{u}_2(t, x) = u(t, x; u_{20}(\cdot + h(s; u_{20}, h_{20}, g)), h_{20} - h(s; u_{20}, h_{20}, g), g).$$

Note that

$$\tilde{u}_1(s, 0) = \tilde{u}_2(s, 0).$$

We must have

$$h_{20} - h(s; u_{20}, h_{20}, g) < h_{10} - h(s; u_{10}, h_{10}, g)$$

and there is a unique  $\xi(0) < h_{20} - h(s; u_{20}, h_{20}, g)$  such that

$$\tilde{u}_2(0, x) \begin{cases} > \tilde{u}_1(0, x) & \text{for } x < \xi(0) \\ < \tilde{u}_1(0, x) & \text{for } \xi(0) < x < h_{20} - h(s; u_{20}, h_{20}, g). \end{cases}$$

Then by the zero number property (see Lemma 4.1),

$$\tilde{u}_2(s, x) > \tilde{u}_1(s, x), \quad -\infty < x < 0.$$

The lemma then follows.  $\square$

Let  $H(x)$  be a  $C^2((-\infty, 0])$  function with  $H'(x) \leq 0$ ,  $H(0) = 0$ ,  $H(x) = 1$  for  $x \leq -1$ . For given  $g \in H(f)$ , let  $u_{0,g}(x)$  and  $u_{n,g}(x)$  be defined by

$$u_{0,g}(x) = \begin{cases} u_g(0), & x < 0 \\ 0, & x = 0 \end{cases}$$

and

$$u_{n,g}(x) = H(nx)u_{0,g}(x).$$

Then

$$u_{n,g}(x) \geq u_{m,g}(x), \quad \forall n \geq m, x \leq 0$$

and

$$u_{n,g}(x) \rightarrow u_{0,g}(x), \quad \forall x \leq 0$$

as  $n \rightarrow \infty$ . By Proposition 3.2, for any  $h_0 \in \mathbb{R}$  and  $n \geq m$ , we have

$$h(t; u_{n,g}(\cdot - h_0), h_0, g) \geq h(t; u_{m,g}(\cdot - h_0), h_0, g) \quad \forall t > 0$$

and

$$u(t, x; u_{n,g}(\cdot - h_0), h_0, g) \geq u(t, x; u_{m,g}(\cdot - h_0), h_0, g) \quad \forall x \leq h(t; u_{m,g}(\cdot - h_0), h_0, g), t \geq 0.$$

Let

$$h(t; u_{0,g}(\cdot - h_0), h_0, g) = \lim_{n \rightarrow \infty} h(t; u_{n,g}(\cdot - h_0), h_0, g) \quad \forall t \geq 0$$

and

$$u(t, x; u_{0,g}(\cdot - h_0), h_0, g) = \begin{cases} \lim_{n \rightarrow \infty} u(t, x; u_{n,g}(\cdot - h_0), h_0, g), & x < h(t; u_{0,g}(\cdot - h_0), h_0, g) \\ 0 & x = h(t; u_{0,g}(\cdot - h_0), h_0, g). \end{cases}$$

Then we have that  $(u(t, x; u_{0,g}(\cdot - h_0), h_0, g), h(t; u_{0,g}(\cdot - h_0), h_0, g))$  is a solution of (4.2) for  $t > 0$  and

$$(u(0, x; u_{0,g}(\cdot - h_0), h_0, g), h(0; u_{0,g}(\cdot - h_0), h_0, g)) = (u_{0,g}(x - h_0), h_0) \quad \forall x \leq h_0.$$

**Lemma 4.3.** *For any given  $g \in H(f)$ ,  $h_{10}, h_{20} \in \mathbb{R}$  and  $u_{20}$  satisfying (4.3) with  $h_0 = h_{20}$  and  $u_{20}(x) < u_g(0)$  for all  $x \leq h_{20}$ , there holds*

$$u(s, x + h(s; u_{0,g}(\cdot - h_{10}), h_{10}, g); u_{0,g}(\cdot - h_{10}), h_{10}, g) \geq u(s, x + h(s; u_{20}, h_{20}, g); u_{20}, h_{20}, g)$$

for all  $x \leq 0$  and  $s \geq 0$ .

*Proof.* First, we note that for any  $n$  large enough,  $u'_{n,g}(x_1) < u'_{20}(x_2)$  for any  $x_1, x_2$  satisfying that  $u_{n,g}(x_1) = u_{20}(x_2)$ . Then by Lemma 4.2,

$$u(s, x + h(s; u_{n,g}(\cdot - h_{10}), h_{10}, g); u_{n,g}(\cdot - h_{10}), h_{10}, g) \geq u(s, x + h(s; u_{20}, h_{20}, g); u_{20}, h_{20}, g)$$

for all  $x \leq 0$ ,  $s \geq 0$ , and  $n \gg 1$ . Letting  $n \rightarrow \infty$ , we have

$$u(s, x + h(s; u_{0,g}(\cdot - h_{10}), h_{10}, g); u_{0,g}(\cdot - h_{10}), h_{10}, g) \geq u(s, x + h(s; u_{20}, h_{20}, g); u_{20}, h_{20}, g)$$

for all  $x \leq 0$  and  $s \geq 0$ . The lemma is thus proved.  $\square$

## 4.2 Basic properties of diffusive KPP equations in fixed unbounded domains

In this section, we presentation some basic properties of solutions of (1.10) and (4.1).

First of all, by the relation between the solutions of (4.1) and (4.2), we have

**Lemma 4.4.** (1) *For given  $u_{01}, u_{02} \in \hat{X}^+$ , if  $u'_{01}(x) \geq 0$ ,  $u'_{02}(x) \geq 0$ , and  $u'_{02}(x_2) > u'_{01}(x_1)$  for any  $x_1, x_2 \geq 0$  satisfying that  $u_{01}(x_1) = u_{02}(x_2)$ , then*

$$u(t, x; u_{01}, g) \leq u(t, x; u_{02}, g) \quad \forall x \geq 0, t \geq 0.$$

(2) *For any  $u_0 \in \hat{X}^+$  with  $u_0(x) < u_g(0)$ , there holds*

$$u(t, x; \tilde{u}_{0,g}, g) \geq u(t, x; u_0, g) \quad \forall x \geq 0, t \geq 0,$$

where  $\tilde{u}_{0,g}(x) = u_{0,g}(-x)$  and  $u(t, x; \tilde{u}_{0,g}, g) = u(t, h(t; u_{0,g}, 0, g) - x; u_{0,g}, 0, g)$ .

*Proof.* (1) It follows directly from Lemma 4.2.

(2) It follows from Lemma 4.3.  $\square$

**Lemma 4.5.** *Consider (4.1). For any  $u_0 \in \hat{X}^+$  with  $u_0'(x) \geq 0$  and  $u_0'(0) > 0$ , then  $u_x(t, x; u_0, g) > 0$  for all  $t > 0$ ,  $x \geq 0$ , and  $g \in H(f)$ .*

*Proof.* First of all, it is easily known that  $u_0(x) > 0$  for  $x > 0$ . By comparison principle for parabolic equations,  $u(t, x; u_0, g) \geq 0$  for all  $t \geq 0$ ,  $x \geq 0$  and  $g \in H(f)$ . Hence

$$u_x(t, 0; u_0, g) \geq 0 \quad \forall t \geq 0, x \geq 0, \text{ and } g \in H(f).$$

Note that  $v(t, x) = u_x(t, x; u_0, g)$  is the solution of

$$\begin{cases} v_t = v_{xx} - \mu u_x(t, 0; u_0, g)v_x(t, x) + [g(t, u(t, x; u_0, g)) \\ \quad + u(t, x; u_0, g)g_u(t, u(t, x; u_0, g))]v(t, x), & 0 < x < \infty \\ v(t, 0) \geq 0 \\ v(0, x) = u_0'(x) \geq 0. \end{cases}$$

Then by comparison principle for parabolic equations again,

$$u_x(t, x; u_0, g) \geq 0 \quad \forall t > 0, x \geq 0, \text{ and } g \in H(f).$$

Next, by Hopf Lemma and strong maximum principle for parabolic equations, we have

$$u_x(t, x; u_0, g) > 0 \quad \forall t > 0, x \geq 0 \text{ and } g \in H(f).$$

$\square$

For given  $u_1, u_2 \in \hat{X}^{++}$  with  $u_1(\cdot) \leq u_2(\cdot)$ , we define a metric,  $\rho(u_1, u_2)$ , between  $u_1$  and  $u_2$  as follows,

$$\rho(u_1, u_2) = \inf\{\ln \alpha \mid \alpha \geq 1, u_2(\cdot) \leq \alpha u_1(\cdot)\}.$$

For given  $u_1, u_2 \in \hat{X}^{++}$  with  $u_i'(0) > 0$  and  $u_i'(x) \geq 0$ , by Lemma 4.5,  $u(t, \cdot; u_i, g) \in \hat{X}^{++}$  for  $t > 0$  and  $g \in H(f)$ .

**Lemma 4.6.** *Consider (4.1). For any  $u_0, v_0 \in \hat{X}^{++}$  with  $u_0(\cdot) \neq v_0(\cdot)$ , if  $u(t, \cdot; u_0, g), u(t, \cdot; v_0, g) \in \hat{X}^{++}$ , and  $u(t, \cdot; u_0, g) \leq u(t, \cdot; v_0, g)$  for all  $t > 0$ , then*

$$\rho(u(t_2, \cdot; u_0, g), u(t_2, \cdot; v_0, g)) \leq \rho(u(t_1, \cdot; u_0, g), u(t_1, \cdot; v_0, g))$$

for all  $0 \leq t_1 < t_2$  and  $g \in H(f)$ . Moreover, if  $\lim_{x \rightarrow \infty} u_0(x) = \lim_{x \rightarrow \infty} v_0(x)$ , then

$$\rho(u(t_2, \cdot; u_0, g), u(t_2, \cdot; v_0, g)) < \rho(u(t_1, \cdot; u_0, g), u(t_1, \cdot; v_0, g)).$$

*Proof.* First, for any  $u_0, v_0 \in \hat{X}^{++}$  with  $u_0(\cdot) \leq v_0(\cdot)$ ,  $u_0(\cdot) \neq v_0(\cdot)$ , there is  $\alpha^* > 1$  such that  $\rho(u_0, v_0) = \ln \alpha^*$  and  $v_0 \leq \alpha^* u_0$ . Let

$$w(t, x) = \alpha^* u(t, x; u_0, g)$$

We then have

$$\begin{aligned} w_t(t, x) &= w_{xx}(t, x) - \mu u_x(t, 0; u_0, g) w_x(t, x) + w(t, x) g(t, u(t, x; u_0, g)) \\ &= w_{xx}(t, x) - \mu u_x(t, 0; u_0, g) w_x(t, x) + w(t, x) g(t, w(t, x)) \\ &\quad + w(t, x) g(t, u(t, x; u_0, g)) - w(t, x) g(t, w(t, x)) \\ &> w_{xx}(t, x) - \mu u_x(t, 0; u_0, g) w_x(t, x) + w(t, x) g(t, w(t, x)) \\ &\geq w_{xx}(t, x) - \mu u_x(t, 0; v_0, g) w_x(t, x) + w(t, x) g(t, w(t, x)) \quad \text{for all } t > 0, \quad x \in \mathbb{R}^+, \end{aligned}$$

and

$$w(t, 0) = 0, \quad \text{for all } t > 0.$$

By comparison principle for parabolic equations, we have

$$u(t, x; v_0, g) \leq \alpha^* u(t, x; u_0, g)$$

for  $t > 0$  and  $x > 0$ . Therefore,

$$\rho(u(t, \cdot; u_0, g), u(t, \cdot; v_0, g)) \leq \rho(u_0, v_0) \quad \text{for all } t \geq 0$$

and then

$$\rho(u(t_2, \cdot; u_0, g), u(t_2, \cdot; v_0, g)) \leq \rho(u(t_1, \cdot; u_0, g), u(t_1, \cdot; v_0, g)) \quad \text{for all } 0 \leq t_1 < t_2.$$

Assume that  $u_\infty = \lim_{x \rightarrow \infty} u_0(x) = \lim_{x \rightarrow \infty} v_0(x)$ . Then for any  $t > 0$ ,

$$\lim_{x \rightarrow \infty} u(t, x; u_0, g) = \lim_{x \rightarrow \infty} u(t, x; v_0, g) = u(t; u_\infty, g), \quad (4.5)$$

where  $u(t; u_\infty, g)$  is the solution of (3.6) with  $u(0; u_\infty, g) = u_\infty$ . Since  $\alpha^* > 1$ ,  $u_\infty \neq \alpha^* u_\infty$ . Hence  $v_0 \neq \alpha^* u_0$ . By Hopf Lemma,

$$u_x(t, 0; v_0, g) < \alpha^* u_x(t, 0; u_0, g). \quad (4.6)$$

By (4.5),

$$\lim_{x \rightarrow \infty} u(t, x; v_0, g) = u(t; u_\infty, g) < \alpha^* u(t; u_\infty, g) = \alpha^* \lim_{x \rightarrow \infty} u(t, x; u_0, g). \quad (4.7)$$

By (4.6)-(4.7), there is  $0 < \beta < 1$  such that

$$u(t, x; v_0, g) \leq \beta \alpha^* u(t, x; u_0, g).$$

It then follows that

$$\rho(u(t, \cdot; u_0, g), u(t, \cdot; v_0, g)) < \rho(u_0, v_0)$$

and then for any  $0 \leq t_1 < t_2$ ,

$$\rho(u(t_2, \cdot; u_0, g), u(t_2, \cdot; v_0, g)) < \rho(u(t_1, \cdot; u_0, g), u(t_1, \cdot; v_0, g)).$$

□

## 5 Semi-Wave Solutions and Proof of Theorem 2.1

In this section, we investigate the semi-wave solutions of (1.9) and prove Theorem 2.1.

We first prove some lemmas.

**Lemma 5.1.** *Let  $g \in H(f)$  be given. There is  $u_0 \in \hat{X}^{++}$  such that  $u_0'(x) \geq 0$  for  $x \geq 0$  and  $\inf_{t \geq 0} u_x(t, 0; u_0, g) > 0$ .*

*Proof.* Let  $\epsilon > 0$  be given such that  $\tilde{\lambda}(a, \epsilon\mu, l) > 0$  for  $l \gg 1$ .

First of all, there is  $K > 0$  such that

$$0 \leq u_x(t, x; u_0, g) \leq K, \quad |u_{xx}(t, x; u_0, g)| \leq K$$

for any  $u_0 \in \hat{X}^{++}$  with  $u_0'(x) \geq 0$  for  $x \geq 0$ ,  $u_0'(x) = 0$  for  $x \geq 1$ , and  $\|u_0\|_{\hat{X}} \ll \min\{\frac{\epsilon}{2}, \frac{\epsilon^2}{4K}\}$ . Fix such a  $u_0$  with  $u_0(\cdot) \not\equiv 0$ .

Observe that  $u_x(t, 0; u_0, g) < \epsilon$  for  $0 < t \ll 1$ . Let

$$t_1 = \sup\{\tau \mid u_x(t, 0; u_0, g) < \epsilon, \quad \forall t \in [0, \tau)\}.$$

Then  $u_x(t, 0; u_0, g) < \epsilon$  for  $t \in (0, t_1)$  and  $u_x(t_1, 0; u_0, g) = \epsilon$  in the case  $t_1 < \infty$ . By comparison principle for parabolic equations,

$$u(t, x; u_0, g) \geq u_\epsilon(t, x; u_0, g) \quad \text{for } 0 \leq t < t_1, \quad (5.1)$$

where  $u_\epsilon(t, x; u_0, g)$  is the solution of (3.11) with  $u_\epsilon(0, x; u_0, g) = u_0(x)$ .

Next, if  $t_1 = \infty$ , by Lemma 3.4, the lemma is proved. Otherwise, note that

$$u_x(t, x; u_0, g) = u_x(t, 0; u_0, g) + \int_0^x u_{xx}(t, y; u_0, g) dy \geq u_x(t, 0; u_0, g) - Kx.$$

Hence for  $0 < x < \frac{\epsilon}{2K}$ ,

$$u_x(t_1, x; u_0, g) \geq \frac{\epsilon}{2}$$

and

$$u(t_1, \frac{\epsilon}{2K}; u_0, g) \geq \frac{\epsilon^2}{4K}.$$

We then have that

$$u_x(t_1, x_1; u_0, g) > u_0'(x_0)$$

for any  $x_0, x_1 \geq 0$  such that  $u(t_1, x_1; u_0, g) = u_0(x_0)$ . By Lemma 4.4, we have

$$u(t + t_1, x; u_0, g) \geq u(t, x; u_0, g \cdot t_1) \quad \text{for } t \geq 0.$$

Similarly, let

$$t_2 = \sup\{\tau \mid u_x(t, 0; u_0, g \cdot t_1) < \epsilon \quad \forall t \in [0, \tau)\}.$$

Then

$$u(t + t_1, x; u_0, g) \geq u(t, x; u_0, g \cdot t_1) \geq u_\epsilon(t, x; u_0, g \cdot t_1) \quad \text{for } 0 \leq t < t_2 \quad (5.2)$$

and in the case  $t_2 < \infty$ ,

$$u(t + t_1 + t_2, x; u_0, g) \geq u(t + t_2, x; u_0, g \cdot t_1) \geq u(t, x; u_0, g \cdot (t_1 + t_2)) \quad \text{for } t \geq 0.$$

Repeating the above process, if  $t_{n-1} < \infty$ , let

$$t_n = \sup\{\tau \mid u_x(t, 0; u_0, g \cdot (t_1 + \cdots + t_{n-1})) < \epsilon \quad \text{for } t \in [0, \tau)\},$$

$n = 1, 2, \dots$ . Then

$$\begin{aligned} u(t + t_1 + \cdots + t_{n-1}, x; u_0, g) &\geq u(t, x; u_0, g \cdot (t_1 + \cdots + t_{n-1})) \\ &\geq u_\epsilon(t, x; u_0, g \cdot (t_1 + \cdots + t_{n-1})) \quad \text{for } 0 \leq t < t_n \end{aligned} \quad (5.3)$$

and in the case  $t_n < \infty$ ,

$$u(t + t_1 + \cdots + t_n, x; u_0, g) \geq u(t, x; u_0, g \cdot (t_1 + \cdots + t_n)) \quad \text{for } t \geq 0.$$

It is not difficult to see that  $\inf_{n \geq 1} t_n > 0$ . Then by (5.1)-(5.3) and Lemma 3.4 again,  $\inf_{t \geq 0} u_x(t, 0; u_0, g) > 0$ .  $\square$

**Lemma 5.2.** *For any  $\epsilon > 0$ , there are  $T^* > 0$  and  $x^* > 0$  such that*

$$|u(t, x; u_0, g) - u_g(t)| < \epsilon$$

for  $t \geq T^*$  and  $x \geq x^*$ , where  $u_0$  is as in Lemma 5.1 with  $u_0(x) \leq u_g(0)$ .

*Proof.* First, note that  $u_{\inf} := \inf_{t \geq 0, x \geq 1} u(t, x; u_0, g) > 0$  and  $u_\infty := \lim_{x \rightarrow \infty} u_0(x) > 0$ . We then have

$$\lim_{x \rightarrow \infty} u(t, x; u_0, g) = u(t; u_\infty, g)$$

where  $u(t; u_\infty, g)$  is the solution of (3.6) with  $u(0; u_\infty, g) = u_\infty$ . Also note that, by Lemma 3.3, for any  $\epsilon > 0$ , any  $\tilde{g} \in H(f)$ , there is  $T^* > 0$  such that

$$|u(t; u_{\inf}, \tilde{g}) - u_{\tilde{g}}(t)| < \epsilon/4$$

for  $t \geq T^*$ .

We claim that there is  $x^* \geq 1$  such that

$$|u(t, x; u_0, g) - u_g(t)| < \epsilon$$

for  $t \geq T^*$  and  $x \geq x^*$ . In fact, assume this is not true, then for any  $n \geq 1$ , there are  $x_n \geq n$  and  $t_n \geq T^*$  such that

$$|u(t_n, x_n; u_0, g) - u_g(t_n)| \geq \epsilon.$$

Let

$$u_n(t, x) = u(t - T^* + t_n, x + x_n; u_0, g).$$

Without loss of generality, assume that

$$g \cdot (t_n - T^*) \rightarrow \tilde{g}, \quad u_n(t, x) \rightarrow \tilde{u}(t, x), \quad u_x(t - T^* + t_n, 0; u_0, g) \rightarrow \tilde{\xi}(t)$$

as  $n \rightarrow \infty$ . Then  $\inf_{t \geq 0, x \in \mathbb{R}} \tilde{u}(t, x) \geq u_{\inf}$  and  $\tilde{u}(t, x)$  satisfies of

$$u_t = u_{xx} - \mu \tilde{\xi}(t) u_x + u \tilde{g}(t, u), \quad x \in \mathbb{R}, \quad t \geq 0. \quad (5.4)$$

Note that  $u(t; u_{\inf}, \tilde{g})$  is also the solution of (5.4) with  $u(0; u_{\inf}, \tilde{g}) = u_{\inf}$ . By comparison principle for parabolic equations, we have

$$\tilde{u}(t, x) \geq u(t; u_{\inf}, \tilde{g}) > u_{\tilde{g}}(t) - \epsilon/4$$

for  $t \geq T^*$  and any  $x \in \mathbb{R}$ . Then for  $n \gg 1$ ,

$$\begin{aligned} u(t_n, x_n; u_0, g) &= u_n(T^*, 0) \\ &\geq \tilde{u}(T^*, 0) - \epsilon/4 \\ &> u_{\tilde{g}}(T^*) - \epsilon/2 \\ &> u_{g \cdot (t_n - T^*)}(T^*) - \epsilon \\ &= u_g(t_n) - \epsilon. \end{aligned}$$

Note that

$$u(t, x; u_0, g) \leq u_g(t) \quad \forall t \geq 0, x \geq 0.$$

We then have

$$|u(t_n, x_n; u_0, g) - u_g(t_n)| < \epsilon.$$

This is a contradiction. The claim is then true and the lemma follows.  $\square$

**Corollary 5.1.** *For any  $\tilde{g} \in H(f)$ , let  $t_n \rightarrow \infty$  be such that  $g \cdot t_n \rightarrow \tilde{g}$  and  $u(t_n, \cdot; u_0, g) \rightarrow \tilde{u}_{\tilde{g}}$ . Then  $u^{**}(t, x; \tilde{g}) = u(t, x; \tilde{u}_{\tilde{g}}, \tilde{g})$  is an entire positive solution of (4.1) with  $g$  being replaced by  $\tilde{g}$  and  $\lim_{x \rightarrow \infty} u^{**}(t, x; \tilde{g}) = u_{\tilde{g}}(t)$  uniformly in  $t \in \mathbb{R}$ .*

*Proof.* It follows from Lemmas 5.1 and 5.2 directly.  $\square$

Let  $\tilde{u}_{0, g \cdot t}(x) = u_{0, g \cdot t}(-x)$  for any  $x \in \mathbb{R}^+$ . Observe that, for any given  $g \in H(f)$  and  $T_2 > T_1 > t > 0$ , we have

$$u(T_2 + t, x; \tilde{u}_{0, g \cdot (-T_2)}, g \cdot (-T_2)) = u(T_1 + t, x; u(T_2 - T_1, \cdot; \tilde{u}_{0, g \cdot (-T_2)}, g \cdot (-T_2)), g \cdot (-T_1)).$$

Then by Lemma 4.4,

$$u(T_2 + t, x; \tilde{u}_{0, g \cdot (-T_2)}, g \cdot (-T_2)) \leq u(T_1 + t, x; \tilde{u}_{0, g \cdot (-T_1)}, g \cdot (-T_1)).$$

Let

$$U^*(t, x; g) = \lim_{T \rightarrow \infty} u(t + T, x; \tilde{u}_{0, g \cdot (-T)}, g \cdot (-T)).$$

Then  $U^*(t, x; g)$  is an entire solution of (4.1),



**Lemma 5.3.** *For any entire positive solution  $v(t, x)$  of (4.1) with  $v(t, x) < u_g(t)$ ,*

$$v(t, x) \leq U^*(t, x; g).$$

*Moreover,  $\lim_{x \rightarrow \infty} U^*(t, x; g) = u_g(t)$  uniformly in  $t \in \mathbb{R}$  and  $g \in H(f)$ .*

*Proof.* First, let  $v(t, x)$  be an entire positive solution of (4.1). By Lemma 4.4,

$$v(t, x) = u(t + T, x; v(-T, \cdot), g \cdot (-T)) \leq u(t + T, x; \tilde{u}_{0, g \cdot (-T)}, g \cdot (-T))$$

for any  $t \in \mathbb{R}$ ,  $t + T > 0$ , and  $x \geq 0$ . Letting  $T \rightarrow \infty$ , we have

$$v(t, x) \leq U^*(t, x; g) \quad \forall x \geq 0.$$

Next, let  $u^{**}(t, x; g)$  be the entire solution in Corollary 5.1. By the above arguments,

$$u^{**}(t, x; g) \leq U^*(t, x; g).$$

Note that  $U^*(t, x; g) \leq u_g(t)$ . Then

$$0 \leq u_g(t) - U^*(t, x; g) \leq u_g(t) - u^{**}(t, x; g) \rightarrow 0$$

as  $x \rightarrow \infty$  uniformly in  $t \in \mathbb{R}$  and  $g \in H(f)$ . □

*Proof of Theorem 2.1.* Let  $\tilde{u}^{**}(t, x) = U^*(t, x; f)$ , we only need to prove  $U^*(t, x; g)$  satisfies the properties in Theorem 2.1 for any  $g \in H(f)$ . Theorem 2.1 then follows.

(1) It suffices to prove that  $U^*(t, x; g)$  is almost periodic in  $t$ .

Note that  $U^*(t, x; g) = U^*(0, x; g \cdot t)$  for any  $t \in \mathbb{R}$  and  $g \in H(f)$ . We claim that  $g \in H(f) \mapsto U^*(0, \cdot; g) \in \hat{X}^{++}$  is continuous. Assume that there is  $g_n \in H(f)$  such that  $g_n \rightarrow g^*$  and

$$U^*(0, \cdot; g_n) \rightarrow \tilde{U}^*(\cdot) \neq U^*(0, \cdot; g^*).$$

Then  $u(t, x; \tilde{U}^*, g^*)$  is an entire solution and

$$U^*(t, x; g^*) \geq u(t, x; \tilde{U}^*, g^*).$$

Note that  $\rho(U^*(t, \cdot; g^*), u(t, \cdot; \tilde{U}^*, g^*))$  is nonincreasing in  $t$ . Let

$$\rho_{-\infty} = \lim_{t \rightarrow -\infty} \rho(U^*(t, \cdot; g^*), u(t, \cdot; \tilde{U}^*, g^*)).$$

Then  $\rho_{-\infty} \neq 0$ . Take a sequence  $s_n \rightarrow -\infty$  such that  $g^* \cdot s_n \rightarrow g^{**}$ ,  $U^*(s_n, \cdot; g^*) \rightarrow U^{**}(\cdot)$ , and  $u(s_n, \cdot; \tilde{U}^*, g^*) \rightarrow \tilde{U}^{**}(\cdot)$ . Then

$$u(t, x; U^{**}, g^{**}) = \lim_{n \rightarrow \infty} U^*(t + s_n, x; g^*)$$

and

$$u(t, x; \tilde{U}^{**}, g^{**}) = \lim_{n \rightarrow \infty} u(t + s_n, x; \tilde{U}^*, g^*).$$

Hence

$$u(t, x; U^{**}, g^{**}) \geq u(t, x; \tilde{U}^{**}, g^{**}).$$

Hence  $\rho(u(t, \cdot; U^{**}, g^{**}), u(t, \cdot; \tilde{U}^{**}, g^{**}))$  is well defined and

$$\rho(u(t, \cdot; U^{**}, g^{**}), u(t, \cdot; \tilde{U}^{**}, g^{**})) = \rho_{-\infty}$$

for all  $t \in \mathbb{R}$ . This implies that  $u(t, \cdot; U^{**}, g^{**}) = u(t, \cdot; \tilde{U}^{**}, g^{**})$  and  $\rho_{-\infty} = 0$ , which is a contradiction. Therefore,  $g \in H(f) \mapsto U^*(0, \cdot; g) \in \hat{X}^{++}$  is continuous. We then have  $U^*(t, \cdot; g) = U^*(0, \cdot; g \cdot t)$  is almost periodic in  $t$ .

(2) Suppose that  $u^{**}(t, x; g)$  is also an almost periodic positive solution of (4.1) and

$$\lim_{x \rightarrow \infty} u^{**}(t, x; g) = u_g(t)$$

uniformly in  $t \in \mathbb{R}$ . Then by Lemma 5.3,

$$U^*(t, x; g) \geq u^{**}(t, x; g).$$

By the almost periodicity, there is  $t_n \rightarrow \infty$  such that  $g \cdot t_n \rightarrow g$  and

$$U^*(t_n, x; g) \rightarrow U^*(0, x; g), \quad u^{**}(t_n, x; g) \rightarrow u^{**}(0, x; g)$$

as  $n \rightarrow \infty$  uniformly in  $x \geq 0$ . It then follows that

$$\rho(u^{**}(t, \cdot; g), U^*(t, \cdot; g)) = \text{constant}$$

and then we must have  $u^{**}(t, x; g) \equiv U^*(t, x; g)$ .

(3) For any bounded positive solution  $u(t, x)$  of (4.1) with  $\liminf_{x \rightarrow \infty} \inf_{t \geq 0} u(t, x) > 0$ , suppose that

$$\lim_{t \rightarrow \infty} [U^*(t, x; g) - u(t, x)] \neq 0$$

then there exist  $t_n \rightarrow \infty$ ,  $u^* \in \hat{X}^{++}$ , such that  $g \cdot t_n \rightarrow g^*$ ,  $U^*(t_n, x; g) \rightarrow U^*(0, x; g^*)$ ,  $u(t_n, x) \rightarrow u^*(x)$  and  $U^*(0, \cdot; g^*) \neq u^*(\cdot)$ . Note that  $U^*(t, \cdot; g^*)$  and  $u(t, \cdot; u^*, g^*)$  exists for all  $t \in \mathbb{R}$ , and by Lemma 5.3 we have

$$U^*(t, x; g^*) \geq u(t, x; u^*, g^*)$$

Then  $\rho(U^*(t, \cdot; g^*), u(t, \cdot; u^*, g^*))$  is well defined and decreases as  $t$  increases. Let

$$\rho_{-\infty} = \lim_{t \rightarrow -\infty} \rho(U^*(t, \cdot; g^*), u(t, \cdot; u^*, g^*))$$

Then  $\rho_{-\infty} > 0$ . By the same arguments in (1), we can get  $\rho_{-\infty} = 0$ , which is a contradiction. Therefore

$$\lim_{t \rightarrow \infty} [U^*(t, x; g) - u(t, x)] = 0$$

uniformly in  $x \geq 0$ .

□

## 6 Spreading Speeds in Diffusive KPP Equations with Free Boundary and Proof of Theorem 2.2

In this section, we consider spreading speeds in spatially homogeneous diffusive KPP equations with free boundary and prove Theorem 2.2.

*Proof of Theorem 2.2.* We divide the proof into four steps. We put  $u(t, x) = u(t, x; u_0, h_0)$  and  $h(t) = h(t; u_0, h_0)$  if no confusion occurs.

**Step 1.** We prove that the unique positive almost periodic solution  $V^*(t)$  of the problem (3.3) satisfies

$$\underline{V}_\epsilon(t) \leq V^*(t) \leq \bar{V}_\epsilon(t)$$

where  $\bar{V}_\epsilon(t)$  and  $\underline{V}_\epsilon(t)$  are, respectively, the unique positive almost periodic solution of

$$V_t = V(f(t, V) + \epsilon) \quad (6.1)$$

and

$$V_t = V(f(t, V) - \epsilon), \quad (6.2)$$

and  $0 < \epsilon \ll 1$ .

Obviously,  $\bar{V}_\epsilon$  and  $\underline{V}_\epsilon$  are, respectively, the supersolution and subsolution of (3.3). Hence, by the comparison principle and uniqueness and stability of almost periodic positive solutions of (3.3), we have

$$\underline{V}_\epsilon(t) \leq V^*(t) \leq \bar{V}_\epsilon(t).$$

Furthermore, for any  $0 < \epsilon \ll 1$ , consider the following two problems

$$\begin{cases} v_t = v_{xx} - \mu v_x(t, 0)v_x(t, x) + v(f(t, v) + \epsilon), & 0 < x < \infty \\ v(t, 0) = 0 \end{cases} \quad (6.3)$$

and

$$\begin{cases} z_t = z_{xx} - \mu z_x(t, 0)z_x(t, x) + z(f(t, z) - \epsilon), & 0 < x < \infty \\ z(t, 0) = 0. \end{cases} \quad (6.4)$$

Using the same arguments as in Theorem 2.1, we know that there exist the unique positive almost periodic solution  $v_\epsilon(t, x)$  of (6.3) and  $z_\epsilon(t, x)$  of (6.4) such that

$$\lim_{x \rightarrow \infty} v_\epsilon(t, x) = \bar{V}_\epsilon(t)$$

and

$$\lim_{x \rightarrow \infty} z_\epsilon(t, x) = \underline{V}_\epsilon(t)$$

uniformly in  $t \in \mathbb{R}$ . Let  $\epsilon \rightarrow 0$ , we can get  $\bar{V}_\epsilon(t)$  and  $\underline{V}_\epsilon(t)$  converge to  $V^*(t)$  uniformly in  $t \in \mathbb{R}$ .

**Step 2.** We prove

$$\overline{\lim}_{t \rightarrow \infty} \frac{h(t)}{t} \leq c^*.$$

By Proposition 2.1,

$$\lim_{t \rightarrow \infty} u(t, x) - V^*(t) = 0 \text{ locally uniformly in } x \geq 0. \quad (6.5)$$

Since  $h_\infty = \infty$ , there exists a  $T > 0$  such that

$$h(T) > l^* \text{ and } u(t + T, l^*) \leq \bar{V}_\epsilon(t + T) \text{ for all } t \geq 0.$$

Let

$$\tilde{u}(t, x) = u(t + T, x + l^*) \text{ and } \tilde{h}(t) = h(t + T) - l^*.$$

We obtain

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + \tilde{u}f(t + T, \tilde{u}) & t > 0, 0 < x < \tilde{h}(t) \\ \tilde{u}(t, 0) = u(t + T, l^*), \tilde{u}(t, \tilde{h}(t)) = 0 & t > 0 \\ \tilde{h}'(t) = -\mu\tilde{u}_x(t, \tilde{h}(t)) & t > 0 \\ \tilde{u}(0, x) = u(T, x + l^*) & 0 < x < \tilde{h}(0). \end{cases}$$

Let  $u^*(t)$  be the unique positive solution of the problem

$$\begin{cases} u_t^* = u^*(f(t, u^*) + \epsilon) & t > T \\ u^*(T) = \max\{\bar{V}_\epsilon, \|\tilde{u}(0, \cdot)\|_\infty\}. \end{cases}$$

Then

$$u^*(t) \geq \bar{V}_\epsilon(t) \text{ for all } t \geq T$$

and Lemma 3.3 tells us that

$$\lim_{t \rightarrow \infty} u^*(t) - \bar{V}_\epsilon(t) = 0.$$

Now we have

$$u^*(T) \geq \tilde{u}(0, x), u^*(t + T) \geq \bar{V}_\epsilon(t + T) \geq \tilde{u}(t, 0), u^*(t + T) \geq 0 = \tilde{u}(t, \tilde{h}(t)) \text{ for } t \geq 0.$$

Hence, we can apply the comparison principle to deduce

$$\tilde{u}(t, x) \leq u^*(t + T) \text{ for } t \geq 0, 0 < x < \tilde{h}(t).$$

As a consequence, there exists  $\bar{T} > T$  such that

$$\tilde{u}(t, x) \leq (1 - \epsilon)^{-1} \bar{V}_\epsilon(t + T) \text{ for } t \geq \bar{T}, 0 \leq x \leq \tilde{h}(t).$$

From the Step 1, we know that there exists  $L > l^*$  such that

$$v_\epsilon(t, x) > (1 - \epsilon) \bar{V}_\epsilon(t) \text{ for } t > 0, x \geq L.$$

We now define

$$\xi(t) = (1 - \epsilon)^{-2} \int_0^t \mu(v_\epsilon)_x(s, 0) ds + L + \tilde{h}(\bar{T}) \text{ for } t \geq 0,$$

$$w(t, x) = (1 - \epsilon)^{-2} v_\epsilon(t, \xi(t) - x) \text{ for } t \geq 0, 0 \leq x \leq \xi(t).$$

Then

$$\begin{aligned} \xi'(t) &= (1 - \epsilon)^{-2} \mu(v_\epsilon)_x(t, 0), \\ -\mu w_x(t, \xi(t)) &= (1 - \epsilon)^{-2} \mu(v_\epsilon)_x(t, 0) \end{aligned}$$

and so we have

$$\xi'(t) = -\mu w_x(t, \xi(t)).$$

Clearly,

$$w(t, \xi(t)) = 0, \quad \xi(T + \bar{T}) \geq L + \tilde{h}(\bar{T}).$$

Moreover, for  $0 < x \leq \tilde{h}(\bar{T})$ ,

$$w(T + \bar{T}, x) = (1 - \epsilon)^{-2} v_\epsilon(T + \bar{T}, \xi(T + \bar{T}) - x) \geq (1 - \epsilon)^{-2} v_\epsilon(T + \bar{T}, L) > (1 - \epsilon)^{-1} \bar{V}_\epsilon(T + \bar{T}) \geq \tilde{u}(\bar{T}, x)$$

and for  $\tilde{h}(\bar{T}) < x < \xi(0)$ ,  $w(T + \bar{T}, x) > 0$ .

And for  $t \geq \bar{T}$ , we have

$$w(t + T, 0) = (1 - \epsilon)^{-2} v_\epsilon(t + T, \xi(t + T)) \geq (1 - \epsilon)^{-2} v_\epsilon(t + T, L) > (1 - \epsilon)^{-1} \bar{V}_\epsilon(t + T) \geq \tilde{u}(t, 0).$$

Direct calculations show that, for  $t \geq \bar{T}$  and  $0 < x < \xi(t)$ , with  $\rho = \xi(t) - x$ ,

$$\begin{aligned} w_t - w_{xx} &= (1 - \epsilon)^{-2} [(v_\epsilon)_t + (v_\epsilon)_\rho \cdot \xi'(t) - (v_\epsilon)_{\rho\rho}] \\ &= (1 - \epsilon)^{-2} [\mu(1 - \epsilon)^{-2} (v_\epsilon)_\rho(t, 0)(v_\epsilon)_\rho(t, \rho) + (v_\epsilon)_t - (v_\epsilon)_{\rho\rho}] \\ &\geq (1 - \epsilon)^{-2} [\mu(v_\epsilon)_\rho(t, 0)(v_\epsilon)_\rho(t, \rho) + (v_\epsilon)_t - (v_\epsilon)_{\rho\rho}] \\ &= (1 - \epsilon)^{-2} v_\epsilon(f(t, v_\epsilon) + \epsilon) \\ &\geq w(f(t, w) + \epsilon). \end{aligned}$$

Hence we can use Lemma 3.5 to conclude that

$$w(t + T, x) \geq \tilde{u}(t, x) \text{ for } t \geq \bar{T}, 0 < x < \tilde{h}(t)$$

$$\xi(t + T) \geq \tilde{h}(t) \text{ for } t \geq \bar{T}.$$

It follows that

$$\begin{aligned} \overline{\lim}_{t \rightarrow \infty} \frac{h(t)}{t} &= \overline{\lim}_{t \rightarrow \infty} \frac{\tilde{h}(t - T) + l^*}{t} \leq \overline{\lim}_{t \rightarrow \infty} \frac{\xi(t)}{t} \\ &= \overline{\lim}_{t \rightarrow \infty} \frac{(1 - \epsilon)^{-2} \int_0^t \mu(v_\epsilon)_x(s, 0) ds + L + \tilde{h}(\bar{T})}{t} \\ &= (1 - \epsilon)^{-2} \lim_{t \rightarrow \infty} \frac{\int_0^t \mu(v_\epsilon)_x(s, 0) ds}{t}. \end{aligned}$$

Note that  $(v_\epsilon)_x(t, 0) \rightarrow \tilde{u}_x^{**}(t, 0)$  as  $\epsilon \rightarrow 0$  uniformly in  $t \in \mathbb{R}$ . Thus,

$$\overline{\lim}_{t \rightarrow \infty} \frac{h(t)}{t} \leq \lim_{t \rightarrow \infty} \frac{\int_0^t \mu \tilde{u}_x^{**}(s, 0) ds}{t} = c^*. \quad (6.6)$$

**Step 3.** We prove

$$\underline{\lim}_{t \rightarrow \infty} \frac{h(t)}{t} \geq c^*.$$

By Lemma 3.3 and Proposition 2.1, we know that there exists a unique positive almost periodic solution  $v^*(t)$  of the problem

$$v_t = vf(t + T, v)$$

and

$$\lim_{t \rightarrow \infty} [\tilde{u}(t, x) - v^*(t)] = 0 \quad (6.7)$$

locally uniformly in  $x \geq 0$ . Using the comparison principle we have

$$v^*(t) \geq \underline{V}_\epsilon(t + T). \quad (6.8)$$

It then follows that  $\underline{\lim}_{t \rightarrow \infty} [v^*(t) - \underline{V}_\epsilon(t + T)] \geq 0$ .

In view of (6.7), we have

$$\underline{\lim}_{t \rightarrow \infty} [\tilde{u}(t, x) - \underline{V}_\epsilon(t + T)] \geq 0 \text{ locally uniformly in } x \geq 0. \quad (6.9)$$

By the same argument as Lemma 5.3 we can get,

$$z_\epsilon(t, x) \leq \underline{V}_\epsilon(t) \text{ for } 0 < x < \infty. \quad (6.10)$$

Due to the (6.9) and (6.10) we can find some  $\tilde{L} > 0$ ,  $\tilde{T} > T$ , and define

$$\eta(t) = (1 - \epsilon)^2 \int_{\tilde{T}+T}^t \mu(z_\epsilon)_x(s, 0) ds + \tilde{L} \text{ for } t \geq \tilde{T} + T,$$

$$w(t, x) = (1 - \epsilon)^2 z_\epsilon(t, \eta(t) - x) \text{ for } t \geq \tilde{T} + T, 0 \leq x \leq \eta(t)$$

such that

$$\tilde{u}(t, 0) \geq w(t + T, 0) \text{ for } t \geq \tilde{T}$$

and

$$\tilde{u}(\tilde{T}, x) \geq w(\tilde{T} + T, x) \text{ for } 0 \leq x \leq \eta(\tilde{T} + T).$$

Then

$$\begin{aligned} \eta'(t) &= (1 - \epsilon)^2 \mu(z_\epsilon)_x(t, 0) \\ -\mu w_x(t, \eta(t)) &= (1 - \epsilon)^2 \mu(z_\epsilon)_x(t, 0) \end{aligned}$$

and so we have

$$\eta'(t) = -\mu w_x(t, \eta(t)).$$

Clearly,

$$w(t, \eta(t)) = 0.$$

Direct calculations show that, for  $t \geq \tilde{T}$  and  $0 < x < \eta(t)$ , with  $\theta = \eta(t) - x$ ,

$$\begin{aligned}
w_t - w_{xx} &= (1 - \epsilon)^2 [(z_\epsilon)_t + (z_\epsilon)_\theta \cdot \eta'(t) - (z_\epsilon)_{\theta\theta}] \\
&= (1 - \epsilon)^2 [\mu(1 - \epsilon)^2 (z_\epsilon)_\theta(t, 0)(z_\epsilon)_\theta(t, \theta) + (z_\epsilon)_t - (z_\epsilon)_{\theta\theta}] \\
&\leq (1 - \epsilon)^2 [\mu(z_\epsilon)_\theta(t, 0)(z_\epsilon)_\theta(t, \theta) + (z_\epsilon)_t - (z_\epsilon)_{\theta\theta}] \\
&= (1 - \epsilon)^2 z_\epsilon(f(t, z_\epsilon) - \epsilon) \\
&\leq w(f(t, w) - \epsilon).
\end{aligned}$$

Hence we can use Lemma 3.6 to conclude that

$$\begin{aligned}
w(t + T, x) &\leq \tilde{u}(t, x) \quad \text{for } t \geq \tilde{T}, 0 < x < \eta(t + T), \\
\eta(t + T) &\leq \tilde{h}(t) \quad \text{for } t \geq \tilde{T}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{h(t)}{t} &= \lim_{t \rightarrow \infty} \frac{\tilde{h}(t - T) + l^*}{t} \geq \lim_{t \rightarrow \infty} \frac{\eta(t)}{t} \\
&= \lim_{t \rightarrow \infty} \frac{(1 - \epsilon)^2 \int_{\tilde{T}+T}^t \mu(z_\epsilon)_x(s, 0) ds + \tilde{h}(\tilde{T})}{t} \\
&= (1 - \epsilon)^2 \lim_{t \rightarrow \infty} \frac{\int_0^t \mu(z_\epsilon)_x(s, 0) ds}{t}.
\end{aligned}$$

Note that  $(z_\epsilon)_x(t, 0) \rightarrow \tilde{u}_x^{**}(t, 0)$  as  $\epsilon \rightarrow 0$  uniformly in  $t \in \mathbb{R}$ . Thus,

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} \geq \lim_{t \rightarrow \infty} \frac{\int_0^t \mu \tilde{u}_x^{**}(s, 0) ds}{t} = c^*. \quad (6.11)$$

Hence, from (6.6) and (6.11) we have

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = c^*.$$

**Step 4.** We prove that for any  $\epsilon > 0$ ,

$$\lim_{t \rightarrow \infty} \max_{x \leq (c^* - \epsilon)t} |u(t, x) - V^*(t)| = 0.$$

By the estimates for  $\tilde{u}(t, x)$  given in Step 2 of the proof, and for any given small  $\delta > 0$ , there exist  $T^\delta > T$  and  $R^\delta > 0$  such that

$$u(t, x + l^*) \leq (1 - \delta)^{-2} v_\delta(t, \xi(t) - x) \quad \text{for } t \geq T^\delta, 0 \leq x \leq \tilde{h}(t)$$

where

$$\xi(t) = (1 - \delta)^{-2} \int_0^t \mu(v_\delta)_x(s, 0) ds + R^\delta$$

and  $v_\delta$  is the unique almost periodic solution of (6.3) with  $\epsilon$  replaced by  $\delta$  and  $\tilde{h}(t) = h(t + T) - l^*$ .

Similarly, by Step 3 of the proof, there exist  $\tilde{T}^\delta, \bar{T}^\delta > T$  and  $\tilde{R}^\delta$  such that

$$u(t, x + l^*) \geq (1 - \delta)^2 z_\delta(t, \eta(t) - x) \quad \text{for } t \geq \tilde{T}^\delta, 0 \leq x \leq \eta(t).$$

where

$$\eta(t) = (1 - \delta)^2 \int_{\tilde{T}^\delta}^t \mu(z_\delta)_x(s, 0) ds + \tilde{R}^\delta$$

and  $z_\delta$  is the unique almost periodic solution of (6.4) with  $\epsilon$  replaced by  $\delta$ .

Since

$$\lim_{\delta \rightarrow 0} (1 - \delta)^{-2} \mu(v_\delta)_x(t, 0) = \lim_{\delta \rightarrow 0} (1 - \delta)^2 \mu(z_\delta)_x(t, 0) = \mu \tilde{u}_x^{**}(t, 0)$$

uniformly for  $t \geq 0$ , for any  $\epsilon > 0$ , we can find  $\delta_\epsilon \in (0, \epsilon)$  small enough and  $T_\epsilon > 0$  such that for all  $t \geq T_\epsilon$ , we have

$$|(1 - \delta_\epsilon)^{-2} \int_0^t \mu(v_{\delta_\epsilon})_x(s, 0) ds - c^* t| < \frac{\epsilon}{2} t$$

and

$$|(1 - \delta_\epsilon)^2 \int_0^t \mu(z_{\delta_\epsilon})_x(s, 0) ds - c^* t| < \frac{\epsilon}{2} t$$

Let  $\bar{R}^\delta = \tilde{R}^\delta - (1 - \delta)^2 \int_0^{\tilde{T}^\delta} \mu(z_\delta)_x(s, 0) ds$ . Choose  $\bar{T}_\epsilon > T_\epsilon$  such that  $\bar{R}^\delta + \frac{\epsilon}{2} t > 0$  for  $t \geq \bar{T}_\epsilon$ . We now fix  $\delta = \delta_\epsilon$  in  $v_\delta$ ,  $z_\delta$ ,  $\xi$  and  $\eta$ . Obviously, for  $t \geq \bar{T}_\epsilon$ ,

$$\xi(t) - x \geq (c^* - \epsilon)t - x + R^{\delta_\epsilon} + \frac{\epsilon}{2} t$$

$$\eta(t) - x \geq (c^* - \epsilon)t - x + \bar{R}^{\delta_\epsilon} + \frac{\epsilon}{2} t$$

By Step 1, we have

$$\lim_{x \rightarrow \infty} z_{\delta_\epsilon}(t, x) = \underline{V}_{\delta_\epsilon}(t) \quad \text{uniformly for } t \in \mathbb{R}$$

where  $\underline{V}_{\delta_\epsilon}(t)$  is the unique positive almost periodic solution of

$$(\underline{V}_{\delta_\epsilon})_t = \underline{V}_{\delta_\epsilon}(f(t, \underline{V}_{\delta_\epsilon}) - \delta_\epsilon)$$

and

$$\lim_{x \rightarrow \infty} v_{\delta_\epsilon}(t, x) = \bar{V}_{\delta_\epsilon}(t) \quad \text{uniformly for } t \in \mathbb{R}$$

where  $\bar{V}_{\delta_\epsilon}(t)$  is the unique positive almost periodic solution of

$$(\bar{V}_{\delta_\epsilon})_t = \bar{V}_{\delta_\epsilon}(f(t, \bar{V}_{\delta_\epsilon}) + \delta_\epsilon)$$

Furthermore, by the same argument as Lemma 5.3 we can find  $R^\epsilon > 0$  such that for  $x \geq R^\epsilon$ ,

$$v_{\delta_\epsilon}(t, x) \leq \bar{V}_{\delta_\epsilon}(t) \quad \text{for all } t \in \mathbb{R}$$

and

$$z_{\delta_\epsilon}(t, x) \geq \underline{V}_{\delta_\epsilon}(t) - \epsilon \quad \text{for all } t \in \mathbb{R}$$



It follows that, if

$$0 \leq x \leq (c^* - \epsilon)t \quad \text{and} \quad t \geq \max\left\{\frac{2(R^\epsilon - \bar{R}^{\delta_\epsilon})}{\epsilon}, \bar{T}_\epsilon, T^{\delta_\epsilon}, \tilde{T}^{\delta_\epsilon}\right\}$$

then

$$u(t, x + l^*) \leq (1 - \delta_\epsilon)^{-2} v_{\delta_\epsilon}(t, \xi(t) - x) \leq (1 - \delta_\epsilon)^{-2} \bar{V}_{\delta_\epsilon}(t)$$

and

$$u(t, x + l^*) \geq (1 - \delta_\epsilon)^2 z_{\delta_\epsilon}(t, \eta(t) - x) \geq (1 - \delta_\epsilon)^2 [\underline{V}_{\delta_\epsilon}(t) - \epsilon]$$

So we take  $T^* = \max\left\{\frac{2(R^\epsilon - \bar{R}^{\delta_\epsilon})}{\epsilon}, \bar{T}_\epsilon, T^{\delta_\epsilon}, \tilde{T}^{\delta_\epsilon}\right\}$ . If  $t \geq T^*$  and  $l^* \leq x \leq (c^* - \epsilon)t$ , we have

$$(1 - \delta_\epsilon)^2 [\underline{V}_{\delta_\epsilon}(t) - \epsilon] \leq u(t, x) \leq (1 - \delta_\epsilon)^{-2} \bar{V}_{\delta_\epsilon}(t)$$

In the view of Step 1, this implies that

$$(1 - \delta_\epsilon)^2 [\underline{V}_{\delta_\epsilon}(t) - \epsilon] - \bar{V}_{\delta_\epsilon}(t) \leq u(t, x) - V^*(t) \leq (1 - \delta_\epsilon)^{-2} \bar{V}_{\delta_\epsilon}(t) - \underline{V}_{\delta_\epsilon}(t)$$

Let

$$I(\epsilon) = \max\{|(1 - \delta_\epsilon)^2 [\underline{V}_{\delta_\epsilon}(t) - \epsilon] - \bar{V}_{\delta_\epsilon}(t)|, |(1 - \delta_\epsilon)^{-2} \bar{V}_{\delta_\epsilon}(t) - \underline{V}_{\delta_\epsilon}(t)|\}$$

Thus,

$$|u(t, x) - V^*(t)| \leq I(\epsilon).$$

By (6.5),

$$\lim_{t \rightarrow \infty} u(t, x) - V^*(t) = 0 \quad \text{uniformly for } x \in [0, l^*]$$

Hence we can find  $\hat{T} > T^*$  such that

$$|u(t, x) - V^*(t)| \leq I(\epsilon) \quad \text{for } t \geq \hat{T} \text{ and } 0 \leq x \leq l^*$$

Finally, we obtain for all  $t \geq \hat{T}$  and  $0 \leq x \leq (c^* - \epsilon)t$ ,

$$|u(t, x) - V^*(t)| \leq I(\epsilon)$$

Let  $\epsilon \rightarrow 0$ , we have  $I(\epsilon) \rightarrow 0$ . So we get

$$\lim_{t \rightarrow 0} \max_{x \leq (c^* - \epsilon)t} |u(t, x) - V^*(t)| = 0.$$

The proof is now complete. □

## 7 Remarks

We have proved the existence of a unique spreading speed  $c^*$  of (1.8) and the existence of a unique time almost periodic positive semi-wave solution of (1.9). It is seen that the spreading speed of (1.8) and the semi-wave solution of (1.9) are closely related. In this section, we give some remarks on the spreading speed of the double fronts free boundary problem (1.11).

First of all, note that the existence and uniqueness results for solutions of (1.8) with given initial data  $(u_0, h_0)$  can be extended to (1.11) using the same arguments as in Section 5 [9], except that we need to modify the transformation in the proof of Theorem 2.1 in [9] such that both boundaries are straightened. In particular, for given  $g_0 < h_0$  and  $u_0$  satisfying

$$\begin{cases} u_0 \in C^2([g_0, h_0], \mathbb{R}^+) \\ u_0(g_0) = u_0(h_0) = 0 \text{ and } u_0 > 0 \text{ in } (g_0, h_0), \end{cases} \quad (7.1)$$

the system (1.11) has a unique global solution  $(u(t, x; u_0, h_0, g_0), h(t; u_0, h_0, g_0), g(t; u_0, h_0, g_0))$  with  $u(0, x; u_0, h_0, g_0) = u_0(x)$ ,  $h(0; u_0, h_0, g_0) = h_0$ ,  $g(0; u_0, h_0, g_0) = g_0$ . Moreover,  $g(t)$  decreases and  $h(t)$  increases as  $t$  increases. Let

$$g_\infty = \lim_{t \rightarrow \infty} g(t; u_0, h_0, g_0) \quad \text{and} \quad h_\infty = \lim_{t \rightarrow \infty} h(t; u_0, h_0, g_0).$$

We next note that, by (H3), there is  $L^* \geq 0$  such that  $\inf_{l \geq L^*} \tilde{\lambda}(a(\cdot), l) > 0$ , where  $a(t) = f(t, 0)$ . By (H3) again, there is a unique time almost periodic and space homogeneous positive solution  $V^*(t)$  of

$$u_t = u_{xx} + u f(t, u) \quad x \in (-\infty, \infty). \quad (7.2)$$

Moreover, for any  $u_0 \in C_{\text{unif}}^b(\mathbb{R}, \mathbb{R}^+)$  with  $\inf_{x \in (-\infty, \infty)} u_0(x) > 0$ ,  $\lim_{t \rightarrow \infty} \|u(t, \cdot; u_0) - V^*(t)\|_\infty = 0$ . The following proposition then follows from [16, Proposition 6.2].

**Proposition 7.1.** *Assume (H1)-(H3). Let  $u_0$  satisfying (7.1) and  $g_0 < h_0$  be given.*

(1) *Either*

(i)  $h_\infty - g_\infty \leq L^*$  and  $\lim_{t \rightarrow +\infty} \|u(t, \cdot; u_0, h_0, g_0)\|_{C([g(t), h(t)])} = 0$  (i.e. vanishing occurs)

or

(ii)  $h_\infty = -g_\infty = \infty$  and  $\lim_{t \rightarrow \infty} [u(t, x; u_0, h_0, g_0) - V^*(t)] = 0$  locally uniformly for  $x \in (-\infty, \infty)$  (i.e. spreading occurs).

(2) *If  $h_0 - g_0 \geq L^*$ , then  $h_\infty = -g_\infty = \infty$ .*

(3) *Suppose  $h_0 - g_0 < L^*$ . Then there exists  $\mu^* > 0$  such that spreading occurs if  $\mu > \mu^*$  and vanishing occurs if  $\mu \leq \mu^*$ .*

We now have

**Proposition 7.2.** For given  $g_0 < h_0$  and  $u_0$  satisfying (7.1), if spreading occurs, then

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \lim_{t \rightarrow \infty} \frac{-g(t)}{t} = c^*,$$

where  $c^*$  is the spreading speed of (1.8).

*Proof.* Let  $g_0 < h_0$  and  $u_0$  satisfying (7.1) be given. Assume that  $g_\infty = -\infty$  and  $h_\infty = \infty$ . Then there are  $T^* > 0$  and  $N^* > 0$  such that

$$-N^*L^* < g(T^*) < -L^* < L^* < h(T^*) < N^*L^*.$$

Without loss of generality, we may assume that  $g_0 < -L^* < L^* < h_0$  and  $u_0(x) > 0$  for  $g_0 < x < h_0$ . Note that there are  $u_0^- \in C([-L^*, L^*], \mathbb{R}^+)$  and  $u_0^+ \in C([-N^*L^*, N^*L^*], \mathbb{R}^+)$  such that

$$\begin{cases} u_0^-(-x) = u_0^-(x) & \text{for } -L^* < x < L^* \\ u_0^-(\pm L^*) = 0 \\ u_0^-(x) < u_0(x) & \text{for } -L^* < x < L^*, \end{cases}$$

and

$$\begin{cases} u_0^+(-x) = u_0^+(x) & \text{for } -N^*L^* < x < N^*L^* \\ u_0^+(\pm N^*L^*) = 0 \\ u_0(x) < u_0^+(x) & \text{for } -N^*L^* < x < N^*L^*. \end{cases}$$

Hence

$$\begin{cases} u(t, x; u_0^-, L^*, -L^*) \leq u(t, x; u_0, h_0, g_0) & \text{for } g(t; u_0^-, L^*, -L^*) < x < h(t; u_0^-, L^*, -L^*) \\ u(t, x; u_0, g_0, h_0) \leq u(t, x; u_0^+, N^*L^*, -N^*L^*) & \text{for } g(t; u_0, h_0, g_0) < x < h(t; u_0, h_0, g_0). \end{cases}$$

Note that

$$\begin{cases} u(t, -x; u_0^-, L^*, -L^*) = u(t, x; u_0^-, L^*, -L^*) \\ h(t; u_0^-, L^*, -L^*) = -g(t; u_0^-, L^*, -L^*) \end{cases}$$

and

$$\begin{cases} u(t, -x; u_0^+, L^*, -L^*) = u(t, x; u_0^+, L^*, -L^*) \\ h(t; u_0^+, N^*L^*, -N^*L^*) = -g(t; u_0^+, N^*L^*, -N^*L^*). \end{cases}$$

Then  $u_x(t, 0; u_0^-, L^*, -L^*) = u_x(t, 0; u_0^+, N^*L^*, -N^*L^*) = 0$ . This together with Theorem 2.2 implies that

$$c^* = \lim_{t \rightarrow \infty} \frac{h(t; u_0^-, L^*, -L^*)}{t} \leq \lim_{t \rightarrow \infty} \frac{h(t; u_0, h_0, g_0)}{t} \leq \lim_{t \rightarrow \infty} \frac{h(t; u_0^+, N^*L^*, -N^*L^*)}{t} = c^*$$

and

$$c^* = \lim_{t \rightarrow \infty} \frac{-g(t; u_0^-, L^*, -L^*)}{t} \geq \lim_{t \rightarrow \infty} \frac{-g(t; u_0, h_0, g_0)}{t} \geq \lim_{t \rightarrow \infty} \frac{-g(t; u_0^+, N^*L^*, -N^*L^*)}{t} = c^*$$

Hence

$$\lim_{t \rightarrow \infty} \frac{h(t; u_0, h_0, g_0)}{t} = \lim_{t \rightarrow \infty} \frac{-g(t; u_0, h_0, g_0)}{t} = c^*.$$

□

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